AN ELEMENTARY TREATISE ON
COORDINATE GEOMETRY OF THREE DIMENSIONS

MACMILLAN AND CO., Limited
LONDON · BOMBAY · CALCUTTA
MELBOURNE

THE MACMILLAN COMPANY
NEW YORK · BOSTON · CHICAGO
DALLAS · SAN FRANCISCO

THE MACMILLAN CO. OF CANADA, Ltd.
TORONTO

AN ELEMENTARY TREATISE

ON

COORDINATE GEOMETRY

OF THREE DIMENSIONS

BY

ROBERT J. T. BELL, M.A., D.Sc.

LECTURER IN MATHEMATICS AND ASSISTANT TO THE PROFESSOR OF MATHEMATICS
IN THE UNIVERSITY OF GLASGOW

MACMILLAN AND CO., LIMITED
ST. MARTIN'S STREET, LONDON

1914

First Edition 1910.
Second Edition 1912, 1914.

PREFACE

The present elementary text-book embodies the course in Solid Coordinate Geometry which, for several years, it has been part of my duties as Lecturer in Mathematics at the University of Glasgow to give to two classes of students.

For the student whose interests lie in the direction of Applied Mathematics, the book aims at providing a fairly complete exposition of the properties of the plane, the straight line, and the conicoids. It is also intended to furnish him with a book of reference which he may consult when his reading on Applied Mathematics demands a knowledge, say, of the properties of curves or of geodesics. At the same time it is hoped that the student of Pure Mathematics may find here a suitable introduction to the larger treatises on the subject and to works on Differential Geometry and the Theory of Surfaces.

The matter has been arranged so that the first ten chapters contain a first course which includes the properties of conicoids as far as confocals. Certain sections of a less elementary character, and all sections and examples that involve the angle- or distance-formulae for oblique axes have been marked with an asterisk, and may be omitted.

Chapter XI. has been devoted to the discussion of the General Equation of the Second Degree. This order of arrangement entails some repetition, but it has compensating advantages. The student who has studied the special forms of the equation finds less difficulty and vagueness in dealing with the general.

I have omitted all account of Homogeneous Coordinates, Tangential Equations, and the method of Reciprocal Polars,

and have included sections on Ruled Surfaces, Curvilinear Coordinates, Asymptotic Lines and Geodesics. It seemed to be more advantageous to make the student acquainted with the new ideas which these sections involve than to exercise him in the application of principles with which his reading in the geometry of two dimensions must have made him to some extent familiar.

In teaching the subject constant recourse has been had to the treatises of Smith, Frost and Salmon, and the works of Carnoy, de Longchamps and Niewenglowski have been occasionally consulted. My obligations to these authors, which are probably much greater than I am aware of, are gratefully acknowledged. I am specially indebted to Resal, whose methods, given in his *Théorie des Surfaces*, I have found very suitable for an elementary course, and have followed in the work of the last two chapters.

The examples are very numerous. Those attached to the sections are for the most part easy applications of the theory or results of the section. Many of these have been constructed to illustrate particular theorems and others have been selected from university examination papers. Some have been taken from the collections of de Longchamps, Koehler, and Mosnat, to whom the author desires to acknowledge his indebtedness.

I have to thank Profs. Jack and Gibson for their kindly interest and encouragement. Prof. Gibson has read part of the work in manuscript and all the proofs, and it owes much to his shrewd criticisms and valuable suggestions. My colleague, Mr. Neil M'Arthur, has read all the proofs and verified nearly all the examples; part of that tedious task was performed by Mr. Thomas M. MacRobert. I tender my cordial thanks to these two gentlemen for their most efficient help. I desire also to thank Messrs. MacLehose for the excellence of their printing work.

ROBT. J. T. BELL.

GLASGOW, *September*, 1910.

PREFACE TO THE SECOND EDITION

In this edition a few alterations have been made, chiefly in the earlier part of the book. One or two sections have been rewritten and additional figures and illustrative examples have been inserted.

R. J. T. B.

June, 1912.

CONTENTS

CHAPTER I

SYSTEMS OF COORDINATES. THE EQUATION TO A SURFACE

CHAPTER II

PROJECTIONS. DIRECTION-COSINES. DIRECTION-RATIOS

CHAPTER III

THE PLANE. THE STRAIGHT LINE. THE VOLUME OF A TETRAHEDRON

CHAPTER IV

CHANGE OF AXES

CHAPTER V

THE SPHERE

CHAPTER VI

THE CONE

CHAPTER VII

THE CENTRAL CONICOIDS. THE CONE. THE PARABOLOIDS

CHAPTER VIII

THE AXES OF PLANE SECTIONS. CIRCULAR SECTIONS

CHAPTER IX

GENERATING LINES

CHAPTER X

CONFOCAL CONICOIDS

CHAPTER XI

THE GENERAL EQUATION OF THE SECOND DEGREE

CHAPTER XII

THE INTERSECTION OF TWO CONICOIDS. SYSTEMS OF CONICOIDS

CHAPTER XIII

CONOIDS. SURFACES IN GENERAL

CHAPTER XIV

CURVES IN SPACE

CHAPTER XV

ENVELOPES. RULED SURFACES

CHAPTER XVI

CURVATURE OF SURFACES

CHAPTER XVII

ASYMPTOTIC LINES—GEODESICS

CHAPTER I.

SYSTEMS OF COORDINATES. THE EQUATION TO A SURFACE.

1. Segments. Two segments AB and CD are said to have the same direction when they are collinear or parallel, and when B is on the same side of A as D is of C. If AB and CD have the same direction, BA and CD have opposite directions. If AB and CD are of the same length and in the same direction they are said to be **equivalent segments.**

2. If A, B, C, ... N, P are any points on a straight line X′OX, and the convention is made that a segment of the straight line is positive or negative according as its direction is that of OX or OX′, then we have the following relations:

$$\text{AB} = -\text{BA}; \quad \text{OA} + \text{AB} = \text{OB}, \quad \text{or} \quad \text{AB} = \text{OB} - \text{OA},$$

$$\text{or} \quad \text{OA} + \text{AB} + \text{BO} = 0;$$

$$\text{OA} + \text{AB} + \text{BC} + \dots \text{NP} = \text{OP}.$$

If x_1, x_2 are the measures of OA and OB, *i.e.* the ratios of OA and OB to any positive segment of unit length, L, then

$$\text{OA} = x_1\text{L}, \quad \text{OB} = x_2\text{L},$$

and

$$\text{AB} = (x_2 - x_1)\,\text{L},$$

or the measure of AB is $x_2 - x_1$.

3. Coordinates. Let X′OX, Y′OY, Z′OZ be any three fixed intersecting lines which are not coplanar, and whose positive directions are chosen to be X′OX, Y′OY, Z′OZ; and let planes through any point in space, P, parallel respectively to the planes YOZ, ZOX, XOY, cut X′X, Y′Y, Z′Z in A, B, C, (fig. 1), then the position of P is known when the segments

OA, OB, OC are given in magnitude and sign. A construction for P would be: cut off from OX the segment OA, draw AN, through A, equivalent to the segment OB, and draw NP, through N, equivalent to the segment OC. OA, OB, OC are known when their measures are known, and these measures are called the **Cartesian coordinates** of P with reference to the coordinate axes X′OX, Y′OY, Z′OZ. The point O is called the origin and the planes YOZ, ZOX, XOY, the coordinate planes. The measure of OA, the segment cut off from OX or OX′ by the plane through P

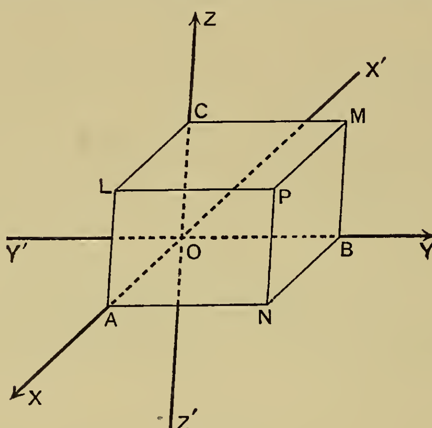

FIG. 1.

parallel to YOZ, is called the x-coordinate of P; the measures of OB and OC are the y- and z-coordinates, and the symbol P, (x, y, z) is used to denote, "the point P whose coordinates are x, y, z." The coordinate planes divide space into eight parts called octants, and the signs of the coordinates of a point determine the octant in which it lies. The following table shews the signs for the eight octants:

Octant	OXYZ	OX′YZ	OX′Y′Z	OXY′Z	OXYZ′	OX′YZ′	OX′Y′Z′	OXY′Z′
x	+	−	−	+	+	−	−	+
y	+	+	−	−	+	+	−	−
z	+	+	+	+	−	−	−	−

It is generally most convenient to choose mutually perpendicular lines as coordinate axes. The axes are then "rectangular," otherwise they are "oblique."

Ex. 1. Sketch in a figure the positions of the points:
$(8, 0, 3)$, $(-2, -1, 5)$, $(-4, -2, 0)$, $(0, 0, -6)$.

Ex. 2. What is the locus of the point, (i) whose x-coordinate is 3, (ii) whose x-coordinate is 2 and whose y-coordinate is -4?

Ex. 3. What is the locus of a point whose coordinates satisfy (i) $x=0$ and $y=0$; (ii) $x=a$ and $y=0$; (iii) $x=a$ and $y=b$; (iv) $z=c$ and $y=b$?

Ex. 4. If $OA=a$, $OB=b$, $OC=c$, (fig. 1), what are the equations to the planes PNBM, PMCL, PNAL? What equations are satisfied by the coordinates of any point on the line PN?

4. Sign of direction of rotation. By assigning positive directions to a system of rectangular axes $X'X$, $Y'Y$, $Z'Z$, we have fixed the positive directions of the normals to the coordinate planes YOZ, ZOX, XOY. Retaining the usual convention made in plane geometry, the positive direction of rotation for a ray revolving about O in the plane XOY is that given by $XYX'Y'$, that is, is counter-clockwise, if the clock dial be supposed to coincide with the plane and front in the positive direction of the normal. Hence to fix the positive direction of rotation for a ray in *any* plane, we have the rule: *if a clock dial is considered to coincide with the plane and front in the positive direction of the normal to the plane, the positive direction of rotation for a ray revolving in the plane is counter-clockwise.* Applying this rule to the other coordinate planes the positive directions of rotation for the planes YOZ, ZOX are seen to be $YZY'Z'$, $ZXZ'X'$.

The positive direction of rotation for a plane can also be found by considering that it is the direction in which a right-handed gimlet or corkscrew has to be turned so that it may move forward in the positive direction of the normal to the plane.

Ex. A plane ABC meets the axes OX, OY, OZ in A, B, C, and ON is the normal from O. If ON is chosen as the positive direction of the normal, and a point P moves round the perimeter of the triangle ABC in the direction ABC, what is the sign of the direction of rotation of NP when OA, OB, OC are (i) all positive, (ii) one negative, (iii) two negative, (iv) all negative?

5. Cylindrical coordinates. If X′OX, Y′OY, Z′OZ, are rectangular axes, and PN is the perpendicular from any point P to the plane XOY, the position of P is determined if ON, the angle XON, and NP are known. The measures of these quantities, u, ϕ, z, are the **cylindrical coordinates** of P. The positive direction of rotation for the plane XOY has been defined, and the direction of a ray originally coincident with OX, and then turned through the given angle ϕ, is the positive direction of ON. In the figure, u, ϕ, z are all positive.

If the Cartesian coordinates of P are x, y, z, those of N are x, y, 0. If we consider only points in the plane XOY, the Cartesian coordinates of N are x, y, and the polar, u, ϕ. Therefore

$$x = u\cos\phi,\ \ y = u\sin\phi;\quad u^2 = x^2 + y^2,\ \tan\phi = y/x.$$

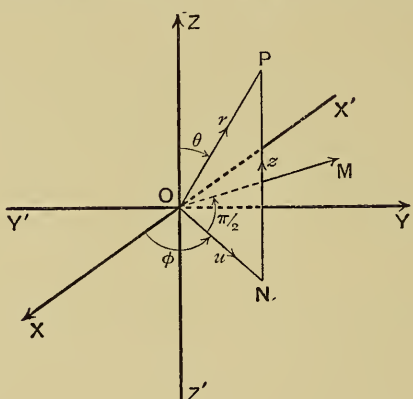

FIG. 2.

6. Polar coordinates. Suppose that the position of the plane OZPN, (fig. 2), has been determined by a given value of ϕ, then we may define the positive direction of the normal through O to the plane to be that which makes an angle $\phi + \pi/2$ with X′OX. Our convention, (§ 4), then fixes the positive direction of rotation for a ray revolving in the plane OZPN. The position of P is evidently determined when, in addition to ϕ, we are given r and θ, the measures

of OP and ∠ZOP. The quantities r, θ, ϕ are the **polar coordinates** of P. The positive direction of OP is that of a ray originally coincident with OZ and then turned in the plane OZPN through the given angle θ. In the figure, OM is the positive direction of the normal to the plane OZPN, and r, θ, ϕ are all positive.

If we consider P as belonging to the plane OZPN and OZ and ON as rectangular axes in that plane, P has Cartesian coordinates z, u, and polar coordinates r, θ. Therefore

$$z=r\cos\theta,\quad u=r\sin\theta;\quad r^2=z^2+u^2,\ \tan\theta=\frac{u}{z}.$$

But if P is (x, y, z), $x=u\cos\phi$, $y=u\sin\phi$.

Whence $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$;

$$r^2=x^2+y^2+z^2,\ \tan\theta=\frac{\pm\sqrt{x^2+y^2}}{z},\ \tan\phi=\frac{y}{x}.$$

Cor. If the axes are rectangular the distance of (x, y, z) from the origin is given by $\sqrt{x^2+y^2+z^2}$.

Ex. 1. Draw figures shewing the positions of the points

$$\left(2, -\frac{\pi}{3}, \frac{\pi}{4}\right),\ \left(3, \frac{2\pi}{3}, -\frac{\pi}{6}\right),\ \left(-3, -\frac{\pi}{4}, -\frac{\pi}{3}\right),\ \left(2, \frac{5\pi}{6}, -\frac{3\pi}{4}\right).$$

What are the Cartesian coordinates of the points?

Ex. 2. Find the polar coordinates of the points $(3, 4, 5)$, $(-2, 1, -2)$, so that r may be positive.

Ans. $\left(5\sqrt{2}, \frac{\pi}{4}, \tan^{-1}\frac{4}{3}\right)$, $\left(3, \frac{\pi}{2}+\tan^{-1}\frac{2\sqrt{5}}{5}, \frac{\pi}{2}+\tan^{-1}2\right)$,

where $\tan^{-1}\frac{4}{3}$, $\tan^{-1}\frac{2\sqrt{5}}{5}$, $\tan^{-1}2$ are acute angles.

Ex. 3. Shew that the distances of the point $(1, 2, 3)$ from the coordinate axes are $\sqrt{13}$, $\sqrt{10}$, $\sqrt{5}$.

Ex. 4. Find (i) the Cartesian, (ii) the cylindrical, (iii) the polar equation of the sphere whose centre is the origin and radius 4.

Ans. (i) $x^2+y^2+z^2=16$, (ii) $u^2+z^2=16$, (iii) $r=4$.

Ex. 5. Find (i) the polar, (ii) the cylindrical, (iii) the Cartesian equation of the right circular cone whose vertex is O, axis OZ, and semivertical angle α.

Ans. (i) $\theta=\alpha$, (ii) $u=z\tan\alpha$, (iii) $x^2+y^2=z^2\tan^2\alpha$.

Ex. 6. Find (i) the cylindrical, (ii) the Cartesian, (iii) the polar equation of the right circular cylinder whose axis is OZ and radius a.

Ans. (i) $u=a$, (ii) $x^2+y^2=a^2$, (iii) $r\sin\theta=a$.

Ex. 7. Find (i) the polar, (ii) the Cartesian equation to the plane through OZ which makes an angle α with the plane ZOX.

Ans. (i) $\phi=\alpha$, (ii) $y=x\tan\alpha$.

7. Change of origin. Let X′OX, Y′OY, Z′OZ; $\alpha'\omega\alpha$, $\beta'\omega\beta$, $\gamma'\omega\gamma$, (fig. 3), be two sets of parallel axes, and let any point P be (x, y, z) referred to the first and (ξ, η, ζ) referred to the second set. Let ω have coordinates a, b, c, referred to OX, OY, OZ. NM is the line of intersection of the planes $\beta\omega\gamma$, XOY, and the plane through P parallel to $\beta\omega\gamma$ cuts $\alpha\omega\beta$ in GH and XOY in KL.

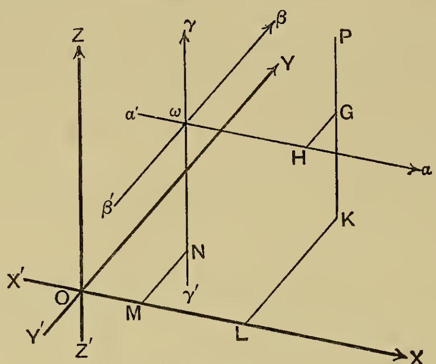

FIG. 3.

Then $$\mathrm{OL}=\mathrm{OM}+\mathrm{ML}=\mathrm{OM}+\omega\mathrm{H},$$

therefore $x=a+\xi$. Similarly, $y=b+\eta$, $z=c+\zeta$;

whence $$\xi=x-a,\ \eta=y-b,\ \zeta=z-c.$$

Ex. 1. The coordinates of (3, 4, 5), (−1 −5, 0), referred to parallel axes through (−2, −3, −7), are (5, 7, 12), (1, −2, 7).

Ex. 2. Find the distance between P, (x_1, y_1, z_1) and Q, (x_2, y_2, z_2), the axes being rectangular.

Change the origin to P, and the coordinates of Q become x_2-x_1, y_2-y_1, z_2-z_1; and the distance is given by

$$\{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2\}^{\frac{1}{2}}.$$

Ex. 3. The axes are rectangular and A, B are the points (3, 4, 5), (−1, 3, −7). A variable point P has coordinates x, y, z. Find the equations satisfied by x, y, z, if (i) PA = PB, (ii) $\mathrm{PA}^2+\mathrm{PB}^2=2k^2$, (iii) $\mathrm{PA}^2-\mathrm{PB}^2=2k^2$.

Ans. (i) $8x+2y+24z+9=0$,

(ii) $2x^2+2y^2+2z^2-4x-14y+4z+109=2k^2$,

(iii) $8x+2y+24z+9+2k^2=0$.

Ex. 4. Find the centre of the sphere through the four points (0, 0, 0), (0, 2, 0), (1, 0, 0), (0, 0, 4). *Ans.* $(\frac{1}{2}, 1, 2)$.

Ex. 5. Find the equation to the sphere whose centre is (0, 1, −1) and radius 2. *Ans.* $x^2+y^2+z^2-2y+2z=2$.

Ex. 6. Prove that $x^2-y^2+z^2-4x+2y+6z+12=0$ represents a right circular cone whose vertex is the point (2, 1, −3), whose axis is parallel to OY and whose semivertical angle is 45°.

Ex. 7. Prove that $x^2+y^2+z^2-2x+4y-6z-2=0$ represents a sphere whose centre is at (1, −2, 3) and radius 4.

8. *To find the coordinates of the point which divides the join of* P, (x_1, y_1, z_1) *and* Q, (x_2, y_2, z_2) *in a given ratio,* $\lambda:1$.

Let R, (x, y, z), (fig. 4), be the point, and let planes through P, Q, R, parallel to the plane YOZ, meet OX in P′, Q′, R′.

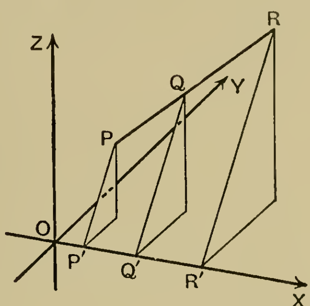

FIG. 4.

Then, since three parallel planes divide any two straight lines proportionally, $P'R':P'Q'=PR:PQ=\lambda:\lambda+1$. Therefore

$$\frac{x-x_1}{x_2-x_1}=\frac{\lambda}{\lambda+1}, \quad \text{and} \quad x=\frac{\lambda x_2+x_1}{\lambda+1}.$$

Similarly, $$y=\frac{\lambda y_2+y_1}{\lambda+1}, \quad z=\frac{\lambda z_2+z_1}{\lambda+1}.$$

These give the coordinates of R for all real values of λ, positive or negative. If λ is positive, R lies between P and Q; if negative, R is on the same side of both P and Q.

Cor. The mid-point of PQ is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$.

Ex. 1. Find the coordinates of the points that divide the join of $(2, -3, 1)$, $(3, 4, -5)$ in the ratios $1:3$, $-1:3$, $3:-2$.

Ans. $\left(\frac{9}{4}, -\frac{5}{4}, -\frac{1}{2}\right)$, $\left(\frac{3}{2}, -\frac{13}{2}, 4\right)$, $(5, 18, -17)$.

Ex. 2. Given that P, $(3, 2, -4)$; Q, $(5, 4, -6)$; R, $(9, 8, -10)$ are collinear, find the ratio in which Q divides PR. Why can the ratio be found by considering the x-coordinates only? *Ans.* $1:2$.

Ex. 3. A, (x_1, y_1, z_1); B, (x_2, y_2, z_2); C, (x_3, y_3, z_3); D, (x_4, y_4, z_4) are the vertices of a tetrahedron. Prove that A′, the centroid of the triangle BCD, has coordinates

$$\frac{x_2+x_3+x_4}{3}, \quad \frac{y_2+y_3+y_4}{3}, \quad \frac{z_2+z_3+z_4}{3}.$$

If B′, C′, D′ are the centroids of the triangles CDA, DAB, ABC, prove that AA′, BB′, CC′, DD′ divide one another in the ratio $3:1$.

Ex. 4. Shew that the lines joining the mid-points of opposite edges of a tetrahedron bisect one another, and that if they be taken for coordinate axes, the coordinates of the vertices can be written (a, b, c), $(a, -b, -c)$, $(-a, b, -c)$, $(-a, -b, c)$.

Ex. 5. Shew that the coordinates of any three points can be put in the form $(a, b, 0)$, $(a, 0, c)$, $(0, b, c)$, a fourth given point being taken as origin.

Ex. 6. The centres of gravity of the tetrahedra ABCD, A′B′C′D′, (Ex. 3), coincide.

Ex. 7. Find the ratios in which the coordinate planes divide the line joining the points $(-2, 4, 7)$, $(3, -5, 8)$. *Ans.* $2:3$, $4:5$, $-7:8$.

Ex. 8. Find the ratios in which the sphere $x^2+y^2+z^2=504$ divides the line joining the points $(12, -4, 8)$, $(27, -9, 18)$. *Ans.* $2:3$, $-2:3$.

Ex. 9. The sphere $x^2+y^2+z^2-2x+6y+14z+3=0$ meets the line joining A, $(2, -1, -4)$; B, $(5, 5, 5)$ in the points P and Q. Prove that $AP:PB=-AQ:QB=1:2$.

Ex. 10. A is the point $(-2, 2, 3)$ and B the point $(13, -3, 13)$. A point P moves so that $3PA=2PB$. Prove that the locus of P is the sphere given by

$$x^2+y^2+z^2+28x-12y+10z-247=0,$$

and verify that this sphere divides AB internally and externally in the ratio $2:3$.

Ex. 11. From the point $(1, -2, 3)$ lines are drawn to meet the sphere $x^2+y^2+z^2=4$, and they are divided in the ratio $2:3$. Prove that the points of section lie on the sphere

$$5x^2+5y^2+5z^2-6x+12y-18z+22=0.$$

9. The equation to a surface. *Any equation involving one or more of the current coordinates of a variable point represents a surface or system of surfaces which is the locus of the variable point.*

The locus of all points whose x-coordinates are equal to a constant α, is a plane parallel to the plane YOZ, and the equation $x=\alpha$ represents that plane. If the equation $f(x)=0$ has roots α_1, α_2, α_3, ... α_n, it is equivalent to the equations $x=\alpha_1$, $x=\alpha_2$, ... $x=\alpha_n$, and therefore represents a system of planes, real or imaginary, parallel to the plane YOZ.

Similarly, $f(y)=0$, $f(z)=0$ represent systems of planes parallel to ZOX, XOY. In the same way, if polar coordinates be taken, $f(r)=0$ represents a system of spheres with a common centre at the origin, $f(\theta)=0$, a system of coaxal right circular cones whose axis is OZ, $f(\phi)=0$, a system of planes passing through OZ.

Consider now the equation $f(x, y)=0$. This equation is satisfied by the coordinates of all points of the curve in the plane XOY whose two-dimensional equation is $f(x, y)=0$.

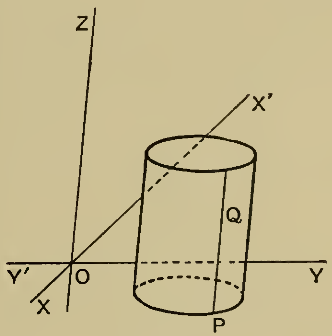

Fig. 5.

Let P, (fig. 5), any point of the curve, have coordinates x_0, y_0, 0. Draw through P a parallel to OZ, and let Q be any point on it. Then the coordinates of Q are x_0, y_0, z_0, and since P is on the curve, $f(x_0, y_0)=0$, thus the coordinates of Q satisfy the equation $f(x, y)=0$. Therefore the coordinates of every point on PQ satisfy the equation and every point on PQ lies on the locus of the equation. But P is any point of the curve, therefore the locus of the equation is the cylinder generated by straight lines drawn

parallel to **OZ** through points of the curve. Similarly, $f(y, z)=0$, $f(z, x)=0$ represent cylinders generated by parallels to **OX** and **OY** respectively.

Ex. What surfaces are represented by (i) $x^2+y^2=a^2$, (ii) $y^2=4ax$, the axes being rectangular?

Two equations are necessary to determine the curve in the plane **XOY**. The curve is on the cylinder whose equation is $f(x, y)=0$ and on the plane whose equation is $z=0$, and hence "the equations to the curve" are $f(x, y)=0$, $z=0$.

Ex. What curves are represented by

(i) $x^2+y^2=a^2$, $z=0$; (ii) $x^2+y^2=a^2$, $z=b$; (iii) $z^2=4ax$, $y=c$?

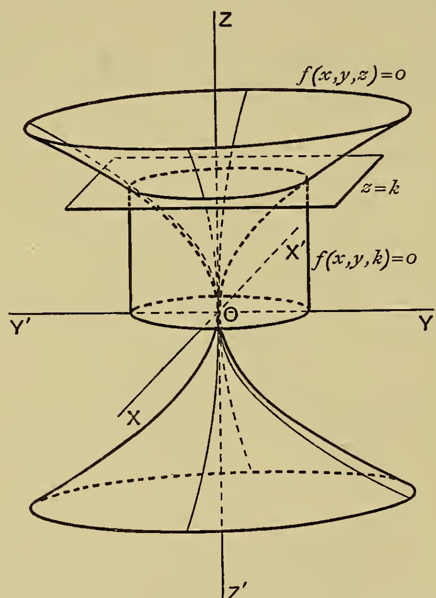

FIG. 6.

(The surface shewn is represented by the equation $a^2x^2+b^2y^2=z^4$.)

Consider now the equation $f(x, y, z)=0$. The equation $z=k$ represents a plane parallel to **XOY**, and the equation $f(x, y, k)=0$ represents, as we have just proved, a cylinder

generated by lines parallel to **OZ**. The equation $f(x, y, k)=0$ is satisfied at all points where $f(x, y, z)=0$ and $z=k$ are simultaneously satisfied, *i.e.* at all points common to the plane and the locus of the equation $f(x, y, z)=0$, and hence $f(x, y, k)=0$ represents the cylinder generated by lines parallel to **OZ** which pass through the common points, (fig. 6). The two equations $f(x, y, k)=0$, $z=k$ represent the curve of section of the cylinder by the plane $z=k$, which is the curve of section of the locus by the plane $z=k$. If, now, all real values from $-\infty$ to $+\infty$ be given to k, the curve $f(x, y, k)=0$, $z=k$, varies continuously and generates a surface. The coordinates of every point on this surface satisfy the equation $f(x, y, z)=0$, for they satisfy, for some value of k, $f(x, y, k)=0$, $z=k$; and any point (x_1, y_1, z_1) whose coordinates satisfy $f(x, y, z)=0$ lies on the surface, for the coordinates satisfy $f(x, y, z_1)=0$, $z=z_1$, and therefore the point is on one of the curves which generate the surface. Hence the equation $f(x, y, z)=0$ represents a surface, and the surface is the locus of a variable point whose coordinates satisfy the equation.

Ex. 1. Discuss the form of the surface represented by

$$x^2/a^2+y^2/b^2+z^2/c^2=1.$$

The section by the plane $z=k$ has equations

$$z=k, \quad x^2/a^2+y^2/b^2=1-k^2/c^2.$$

The section is therefore a real ellipse if $k^2<c^2$, is imaginary if $k^2>c^2$, and reduces to a point if $k^2=c^2$. The surface is therefore generated by a variable ellipse whose plane is parallel to **XOY** and whose centre is on **OZ**. The ellipse increases from a point in the plane $z=-c$ to the ellipse in the plane **XOY** which is given by $x^2/a^2+y^2/b^2=1$, and then decreases to a point in the plane $z=c$. The surface is the *ellipsoid*, (fig. 29).

Ex. 2. What surfaces are represented by the equations, referred to rectangular axes,

$$\text{(i) } x^2+y^2+z^2=a^2, \quad \text{(ii) } x^2+y^2=2az?$$

Ex. 3. Discuss the forms of the surfaces

$$\text{(i) } \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1, \quad \text{(ii) } \frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

(i) The hyperboloid of one sheet (fig. 30).

(ii) The hyperboloid of two sheets (fig. 31).

Ex. 4. What loci are represented by

(i) $f(u)=0$, (ii) $f(z)=0$, (iii) $f(r, \theta)=0$,

(iv) $f(\theta, \phi)=0$, (v) $f(r, \phi)=0$, (vi) $f(u, \phi)=0$?

Ans. (i) A system of coaxial right cylinders; (ii) a system of planes parallel to XOY; (iii) the surface of revolution generated by rotating the curve in the plane ZOX whose polar equation is $f(r, \theta)=0$ about the z-axis; (iv) a cone whose vertex is at O; (v) a surface generated by circles whose planes pass through OZ and whose dimensions vary as the planes rotate about OZ; (vi) a cylinder whose generators are parallel to OZ, and whose section by the plane $z=0$ is the curve $f(u, \phi)=0$.

10. The equations to a curve. The two equations $f_1(x, y, z)=0$, $f_2(x, y, z)=0$ represent the curve of intersection of the two surfaces given by $f_1(x, y, z)=0$ and $f_2(x, y, z)=0$. If we eliminate one of the variables, z,

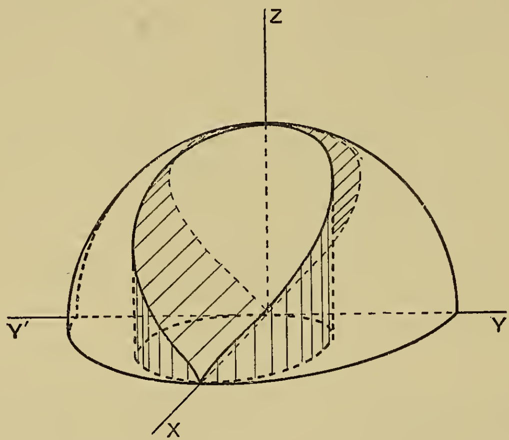

FIG. 7.

Fig. 7 shews part of the curve of intersection of the sphere $x^2+y^2+z^2=a^2$ and the right circular cylinder $x^2+y^2=ax$. The cylinder which projects the curve on the plane $x=0$ is also shewn. Its equation is $a^2(y^2-z^2)+z^4=0$. The projection of the curve on the plane ZOX is the parabola whose equations are $y=0$, $z^2=a(a-x)$.

say, between the two equations, we obtain an equation, $\phi(x, y)=0$, which represents a cylinder whose generators are parallel to OZ. If any values of x, y, z satisfy $f_1(x, y, z)=0$ and $f_2(x, y, z)=0$, they satisfy $\phi(x, y)=0$, and hence the cylinder passes through the curve of intersection of the

surfaces. If the axes are rectangular $\phi(x, y)=0$ represents the cylinder which projects orthogonally the curve of intersection on the plane XOY, and the equations to the projection are $\phi(x, y)=0$, $z=0$.

Ex. 1. If the axes are rectangular, what loci are represented by (i) $x^2+y^2=a^2$, $z^2=b^2$; (ii) $x^2+y^2+z^2=a^2$, $y^2=4az$; (iii) $x^2+y^2=a^2$, $x^2=b^2$, $(a^2>b^2)$?

Ex. 2. Find the equations to the cylinders with generators parallel to OX, OY, OZ, which pass through the curve of intersection of the surfaces represented by $x^2+y^2+2z^2=12$, $x-y+z=1$.

Ans. $2y^2-2yz+3z^2+2y-2z-11=0$, $2x^2+2xz+3z^2-2x-2z-11=0$, $3x^2-4xy+3y^2-4x+4y-10=0$.

11. Surfaces of revolution. Let P, $(0, y_1, z_1)$, (fig. 8), be any point on the curve in the plane YOZ whose Cartesian equation is $f(y, z)=0$. Then

$$f(y_1, z_1)=0 \quad \text{.........................(1)}$$

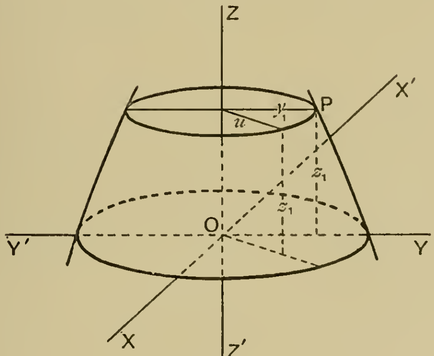

FIG. 8.

The rotation of the curve about OZ produces a surface of revolution. As P moves round the surface, z_1, the z-coordinate of P remains unaltered, and u, the distance of P from the z-axis, is always equal to y_1. Therefore, by (1), the cylindrical coordinates of P satisfy the equation $f(u, z)=0$. But P is any point on the curve, or surface, and therefore the cylindrical equation to the surface is $f(u, z)=0$. Hence the Cartesian equation to the surface is $f(\sqrt{x^2+y^2}, z)=0$.

Since the distance of the point (x, y, z) from the y-axis is $\sqrt{z^2+x^2}$, it follows as before that the equation to the surface formed by rotating the curve $f(y, z)=0$, $x=0$ about OY is $f(y, \sqrt{z^2+x^2})=0$, and similarly $f(\sqrt{y^2+z^2}, x)=0$ represents a surface of revolution whose axis is OX.

Ex. 1. The equation $x^2+y^2+z^2=a^2$ represents the sphere formed by the revolution of the circle $x^2+y^2=a^2$, $z=0$, about OX or OY.

Ex. 2. The surface generated by the revolution of the parabola $y^2=4ax$, $z=0$, about its axis has equation $y^2+z^2=4ax$; about the tangent at the vertex, equation $y^4=16a^2(z^2+x^2)$.

Ex. 3. The surfaces generated by rotating the ellipse $x^2/a^2+y^2/b^2=1$, $z=0$, about its axes are given by $\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1$, $\frac{x^2+z^2}{a^2}+\frac{y^2}{b^2}=1$.

Ex. 4. Find the equations to the cones formed by rotating the line $z=0$, $y=2x$ about OX and OY.

Ans. $4x^2-y^2-z^2=0$, $4x^2-y^2+4z^2=0$.

Ex. 5. Find the equation to the surface generated by the revolution of the circle $x^2+y^2+2ax+b^2=0$, $z=0$, about the y-axis.

Ans. $(x^2+y^2+z^2+b^2)^2=4a^2(x^2+z^2)$.

Ex. 6. Sketch the forms of the surfaces:

(i) $(y^2+z^2)(2a-x)=x^3$, (ii) $r^2=a^2\cos 2\theta$, (iii) $u^2=2cz$.

The surfaces are generated by rotating (i) the curve $y^2(2a-x)=x^3$ about OX; (ii) the lemniscate in the plane ZOX, $r^2=a^2\cos 2\theta$, about OZ; (iii) the parabola in the plane YOZ, $y^2=2cz$, about OZ.

Ex. 7. Prove that the locus of a point, the sum of whose distances from the points $(a, 0, 0)$, $(-a, 0, 0)$ is constant, $(2k)$, is the ellipsoid of revolution $\frac{x^2}{k^2}+\frac{y^2+z^2}{k^2-a^2}=1$.

CHAPTER II.

PROJECTIONS.

12. The angle that a given directed line OP makes with a second directed line OX we shall take to be the smallest angle generated by a variable radius turning in the plane XOP from the position OX to the position OP. The sign of the angle is determined by the usual convention. Thus, in figures 9 and 10, θ_1 is the positive angle, and θ_2 the negative angle that OP makes with OX.

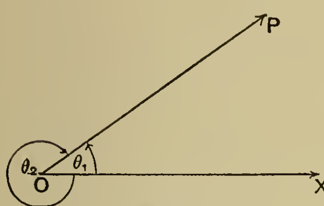

FIG. 9.

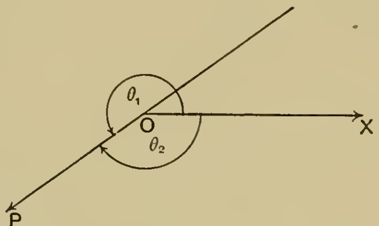

FIG. 10.

13. Projection of a segment. *If* AB *is a given segment and* A′, B′ *are the feet of the perpendiculars from* A, B *to a given line* X′X, *the segment* A′B′ *is the projection of the segment* AB *on* X′X.

From the definition it follows that the projection of BA is B′A′, and therefore that the projections of AB and BA differ only in sign.

It is evident that A′B′ is the intercept made on X′X by the planes through A and B normal to X′X, and hence *the projections of equivalent segments are equivalent segments.*

14. *If* AB *is a given segment of a directed line* MN *whose positive direction,* MN, *makes an angle* θ *with a*

given line X′X, *the projection of* AB *on* X′X *is equal to* AB . cos θ.

In figures 11 and 12, AB is positive, in figures 13 and 14, AB is negative.

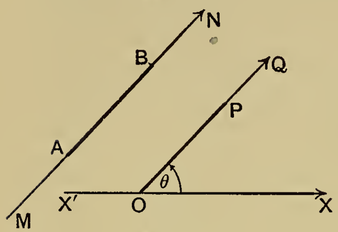

FIG. 11.

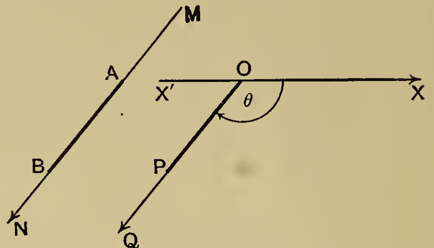

FIG. 12.

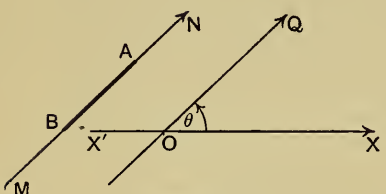

FIG. 13.

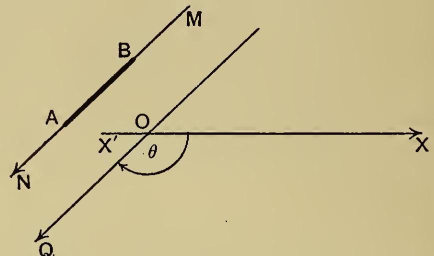

FIG. 14.

Draw OQ from O in the same direction as MN. *If* AB *is positive,* cut off OP, the segment equivalent to AB; then

the projection of AB = the projection of OP,
= OP . cos θ, (by the definition
= AB . cos θ. of cosine),

If AB *is negative,* BA is positive, and therefore

the projection of BA = BA . cos θ,
i.e. −(the projection of AB) = −AB . cos θ,
i.e. the projection of AB = AB . cos θ.

15. *If* A, B, C, ... M, N *are any* n *points in space, the sum of the projections of* AB, BC, ... MN, *on any given line* X′X *is equal to the projection of the straight line* AN *on* X′X.

Let the feet of the perpendiculars from A, B, ... M, N, to X′X be A′, B′, ... M′, N′. Then, (§ 2),

$$A'B' + B'C' + \dots M'N' = A'N',$$

which proves the proposition.

16. The angle between two planes we shall take to be the angle that the positive direction of a normal to one makes with the positive direction of a normal to the other.

17. Projection of a closed plane figure. *If the projections of three points* A, B, C *on a given plane are* A′, B′, C′, *then* $\triangle A'B'C' = \cos\theta \triangle ABC$, *where* θ *is the angle between the planes* ABC, A′B′C′.

Consider first the areas ABC, A′B′C′ without regard to sign.

(i) If the planes ABC, A′B′C′ are parallel, the equation $\triangle A'B'C' = \cos\theta \triangle ABC$ is obviously true.

(ii) If one side of the triangle ABC, say BC, is parallel to the plane A′B′C′, let AA′ meet the plane through BC parallel to the plane A′B′C′ in A_2, (fig. 15). Draw A_2D at right angles to BC, and join AD. Then BC is at right angles to

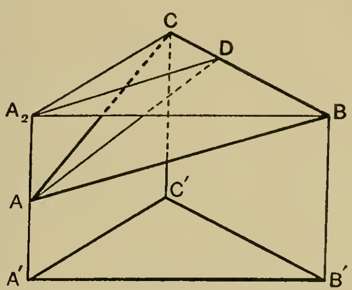

FIG. 15.

A_2D and AA_2, and therefore BC is normal to the plane AA_2D, and therefore at right angles to AD. Hence the angle A_2DA is equal to θ, or its supplement.

But $$\triangle A'B'C' = \triangle A_2BC,$$

and $$\triangle A_2BC : \triangle ABC = A_2D : AD = \cos \angle A_2DA;$$

therefore $$\triangle A'B'C' = \cos\theta \triangle ABC.$$

(iii) If none of the sides of the triangle ABC is parallel to the plane A′B′C′, draw lines through A, B, C parallel to the line of intersection of the planes ABC, A′B′C′. These lines lie in the plane ABC and are parallel to the plane A′B′C′, and one of them, that through A, say, will cut the

opposite side, BC, of the triangle ABC, internally. And therefore the triangle ABC can always be divided by a line through a vertex into two triangles, with a common side parallel to the given plane A′B′C′, and hence, by (ii), $\triangle$A′B′C′$=\cos\theta\,\triangle$ABC.

Suppose now that the areas ABC, A′B′C′ are considered positive or negative according as the directions of rotation given by ABC, A′B′C′ are positive or negative. Then, applying the convention of § 4 to figures 16 and 17, we

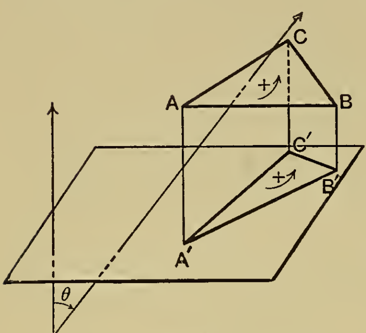

Fig. 16.

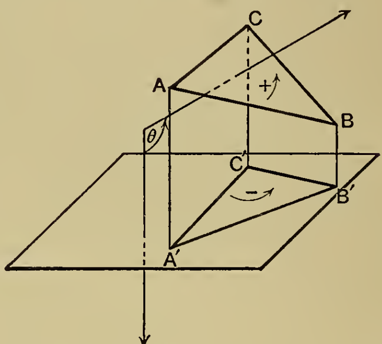

Fig. 17.

see that if $\cos\theta$ is positive, the directions of rotation ABC, A′B′C′ have the same sign, and that if $\cos\theta$ is negative, they have opposite signs. That is, the areas have the same sign if $\cos\theta$ is positive, and opposite signs if $\cos\theta$ is negative. Hence the equation $\triangle$A′B′C′$=\cos\theta\,\triangle$ABC is true for the signs as well as the magnitudes of the areas.

18. *If* A, B, C, ... N *are any coplanar points and* A′, B′, C′, ... N′ *are their projections on any given plane, then*

$$\textit{area}\ \mathrm{A'B'C'\ldots N'} : \textit{area}\ \mathrm{ABC\ldots N} = \cos\theta,$$

where θ is the angle between the planes.

Let O be any point of the plane ABC ... N, and O′ be its projection on the plane A′B′C′ ... N′.

Then area ABC ... N $= \triangle$ OAB $+ \triangle$ OBC $+ \ldots \triangle$ ONA,

and area A′B′C′ ... N′ $= \triangle$ O′A′B′ $+ \triangle$ O′B′C′ $+ \ldots \triangle$ O′N′A′.

But $\triangle$ O′A′B′$=\cos\theta\,\triangle$OAB, etc., and therefore the result follows.

19. *If* A_0 *is the area of any plane curve and* A *is the area of its projection on any given plane,* $A = \cos\theta \,.\, A_0$, *where* θ *is the angle between the planes.*

For A_0 is the limit, as n tends to infinity, of the area of an inscribed n-gon, and A is the limit of the area of the projection of the n-gon, and, by § 18, the ratio of these areas is $\cos\theta$.

Ex. 1. AA' is a diameter of a given circle, and P is a plane through AA' making an angle θ with the plane of the circle. If B is any point on the circle and B' is its projection on the plane P, the perpendiculars from B and B' to AA' are in the constant ratio $1 : \cos\theta$, and the projection is therefore a curve such that its ordinate to AA' is in a constant ratio to the corresponding ordinate of the circle; that is, the projection is an ellipse whose major axis is AA' and whose auxiliary circle is equal to the given circle. The minor axis is $\cos\theta \,.\, AA'$; therefore if $AA' = 2a$ and $\cos\theta = b/a$, the minor axis is $2b$. By § 19, the area of the ellipse $= \cos\theta \,.\, \pi a^2 = \pi ab$.

Ex. 2. Find the area of the section of the cylinder $16x^2 + 9y^2 = 144$ by a plane whose normal makes an angle of $60°$ with OZ. *Ans.* 24π.

DIRECTION-COSINES.

20. If α, β, γ are the angles that a given directed line makes with the positive directions $X'OX$, $Y'OY$, $Z'OZ$ of the coordinate axes, $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the **direction-cosines** of the line.

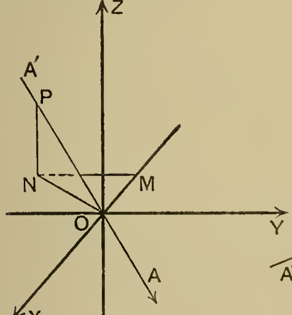

FIG. 18.

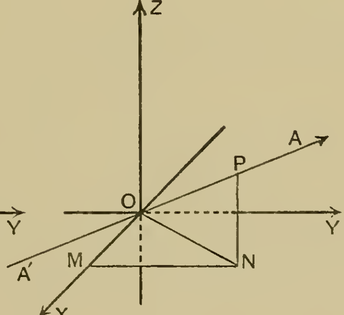

FIG. 19.

21. Direction-cosines referred to rectangular axes. Let $A'OA$ be the line through O which has direction-cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$. Let P, (x, y, z) be any point on $A'OA$,

and OP have measure r. In fig. 18, r is positive; in fig. 19, r is negative. Draw PN perpendicular to the plane XOY, and NM in the plane XOY, perpendicular to OX. Then the measures of OM, MN, NP are x, y, z respectively. Since OM is the projection of OP on OX,

$$x = r\cos\alpha, \text{ and similarly, } y = r\cos\beta,\ z = r\cos\gamma. \quad\ldots\ldots(1)$$

Again the projection of OP on any line is equal to the sum of the projections of OM, MN, NP, and therefore, projecting on OP, we obtain

$$r = x\cos\alpha + y\cos\beta + z\cos\gamma. \quad\ldots\ldots\ldots\ldots(2)$$

But $x/r = \cos\alpha$, $y/r = \cos\beta$, $z/r = \cos\gamma$; therefore

$$1 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma. \quad\ldots\ldots\ldots\ldots(3)$$

This is the formula in three dimensions which corresponds to $\cos^2\theta + \sin^2\theta = 1$ in plane trigonometry.

Cor. 1. By substituting for $\cos\alpha$, $\cos\beta$, $\cos\gamma$ in (2) or (3), we obtain $r^2 = x^2 + y^2 + z^2$, (cf. § 6, Cor.).

Cor. 2. If (x, y, z) is any point on the line through O whose direction-cosines are $\cos\alpha$, $\cos\beta$, $\cos\gamma$, we have, by (1),

$$\frac{x}{\cos\alpha} = \frac{y}{\cos\beta} = \frac{z}{\cos\gamma},\ (= r).$$

Cor. 3. If (x, y, z) is any point on the line through (x_1, y_1, z_1) whose direction-cosines are $\cos\alpha$, $\cos\beta$, $\cos\gamma$, by changing the origin we obtain

$$\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\cos\beta} = \frac{z - z_1}{\cos\gamma}.$$

Ex. 1. Prove that $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$.

Ex. 2. If P is the point (x_1, y_1, z_1), prove that the projection of OP on a line whose direction-cosines are l_1, m_1, n_1 is $l_1x_1 + m_1y_1 + n_1z_1$.

The projection of OP = projn. of OM + projn. of MN
+ projn. of NP, (figs. 18, 19),

$$= l_1x_1 + m_1y_1 + n_1z_1.$$

Ex. 3. If P, Q are the points (x_1, y_1, z_1), (x_2, y_2, z_2), prove that the projection of PQ on a line whose direction-cosines are l_1, m_1, n_1 is

$$l_1(x_2 - x_1) + m_1(y_2 - y_1) + n_1(z_2 - z_1).$$

(Change the origin to P and apply Ex. 2.)

Ex. 4. The projections of a line on the axes are 2, 3, 6. What is the length of the line? *Ans.* 7.

Ex. 5. A plane makes intercepts **OA, OB, OC**, whose measures are a, b, c, on the axes **OX, OY, OZ**. Find the area of the triangle **ABC**.

Let the positive direction of the normal from **O** to the plane **ABC** have direction-cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$, and let Δ denote the area **ABC**. Then since $\triangle$**OBC** is the projection of $\triangle$**ABC** on the plane **YOZ**, $\cos\alpha\,.\,\Delta=\frac{1}{2}bc$, and similarly, $\cos\beta\,.\,\Delta=\frac{1}{2}ca$, $\cos\gamma\,.\,\Delta=\frac{1}{2}ab$. Therefore, since

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1,\quad \Delta=\tfrac{1}{2}\{b^2c^2+c^2a^2+a^2b^2\}^{\frac{1}{2}}.$$

Ex. 6. Find the areas of the projections of the curve $x^2+y^2+z^2=25$, $x+2y+2z=9$ on the coordinate planes, and having given that the curve is plane, find its area.

(Cf. Ex. 2, § 10.) *Ans.* $16\pi/3$, $32\pi/3$, $32\pi/3$; 16π.

22. If a, b, c are given proportionals to the direction-cosines of a line, the actual direction-cosines are found from the relations

$$\frac{\cos\alpha}{a}=\frac{\cos\beta}{b}=\frac{\cos\gamma}{c}=\frac{\sqrt{\cos^2\alpha+\cos^2\beta+\cos^2\gamma}}{\sqrt{a^2+b^2+c^2}}=\frac{\pm 1}{\sqrt{a^2+b^2+c^2}}.$$

If **P** is the point (a, b, c) and the direction-cosines of the directed line **OP** are $\cos\alpha$, $\cos\beta$, $\cos\gamma$, then, since **OP** is positive and equal to $\sqrt{a^2+b^2+c^2}$,

$$\cos\alpha=\frac{a}{\mathsf{OP}}=\frac{a}{\sqrt{a^2+b^2+c^2}},\quad \cos\beta=\frac{b}{\sqrt{a^2+b^2+c^2}},$$

$$\cos\gamma=\frac{c}{\sqrt{a^2+b^2+c^2}}.$$

The direction-cosines of **PO** are

$$\frac{-a}{\sqrt{a^2+b^2+c^2}},\quad \frac{-b}{\sqrt{a^2+b^2+c^2}},\quad \frac{-c}{\sqrt{a^2+b^2+c^2}}.$$

Ex. 1. Find the direction-cosines of a line that makes equal angles with the axes.

Ans. $\cos\alpha=\cos\beta=\cos\gamma=\pm 1/\sqrt{3}$; (whence the acute angles which the line makes with the axes are equal to 54° 44′).

Ex. 2. **P** and **Q** are $(2, 3, -6)$, $(3, -4, 5)$. Find the direction-cosines of **OP, OQ, PO**.

Ans. $\frac{2}{7}, \frac{3}{7}, \frac{-6}{7}$; $\frac{3}{5\sqrt{2}}, \frac{-4}{5\sqrt{2}}, \frac{1}{\sqrt{2}}$; $\frac{-2}{7}, \frac{-3}{7}, \frac{6}{7}$.

Ex. 3. If **P, Q** are (x_1, y_1, z_1), (x_2, y_2, z_2) the direction-cosines of **PQ** are proportional to x_2-x_1, y_2-y_1, z_2-z_1.

Ex. 4. If P, Q are (2, 3, 5), (−1, 3, 2), find the direction-cosines of PQ.
Ans. $\frac{-1}{\sqrt{2}}$, 0, $\frac{-1}{\sqrt{2}}$.

Ex. 5. If P, Q, R, S are the points (3, 4, 5), (4, 6, 3), (−1, 2, 4), (1, 0, 5), find the projection of RS on PQ. *Ans.* $-\frac{4}{3}$.

Ex. 6. If P, Q, R, S are the points (2, 3, −1), (3, 5, −3), (1, 2, 3), (3, 5, 7), prove by projections that PQ is at right angles to RS.

23. The angle between two lines. *If* OP *and* OQ *have direction-cosines* $\cos\alpha$, $\cos\beta$, $\cos\gamma$; $\cos\alpha'$, $\cos\beta'$, $\cos\gamma'$, *and* θ *is the angle that* OP *makes with* OQ,

$$\cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'.$$

If, as in § 21, P is (x, y, z) and the measure of OP is r, projecting OP and OM, MN, NP on OQ, we obtain

$$r\cos\theta = x\cos\alpha' + y\cos\beta' + z\cos\gamma'.$$

But $$x = r\cos\alpha,\quad y = r\cos\beta,\quad z = r\cos\gamma;$$

therefore $$\cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'.$$

Cor. 1. We have the identity

$$(l^2+m^2+n^2)(l'^2+m'^2+n'^2)-(ll'+mm'+nn')^2$$
$$\equiv (mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2.$$

(This identity is known as *Lagrange's identity.* We shall frequently find it advantageous to apply it.)

Hence

$$\sin^2\theta = (\cos^2\alpha+\cos^2\beta+\cos^2\gamma)(\cos^2\alpha'+\cos^2\beta'+\cos^2\gamma')$$
$$-(\cos\alpha\cos\alpha'+\cos\beta\cos\beta'+\cos\gamma\cos\gamma')^2,$$
$$=(\cos\beta\cos\gamma'-\cos\gamma\cos\beta')^2+(\cos\gamma\cos\alpha'-\cos\alpha\cos\gamma')^2$$
$$+(\cos\alpha\cos\beta'-\cos\beta\cos\alpha')^2.$$

Cor. 2. If θ is an angle between the lines whose direction-cosines are proportional to a, b, c; a', b', c',

$$\cos\theta = \frac{\pm(aa'+bb'+cc')}{\sqrt{a^2+b^2+c^2}\sqrt{a'^2+b'^2+c'^2}},$$

and $$\sin\theta = \frac{\pm\sqrt{(bc'-b'c)^2+(ca'-c'a)^2+(ab'-a'b)^2}}{\sqrt{a^2+b^2+c^2}\sqrt{a'^2+b'^2+c'^2}}.$$

Cor. 3. If the lines are at right angles,

$$\cos\alpha\cos\alpha'+\cos\beta\cos\beta'+\cos\gamma\cos\gamma' = 0,\ \text{or}\ aa'+bb'+cc' = 0.$$

Cor. 4. If the lines are parallel,

$$\cos\beta\cos\gamma' - \cos\gamma\cos\beta' = 0, \quad \cos\gamma\cos\alpha' - \cos\alpha\cos\gamma' = 0,$$

and
$$\cos\alpha\cos\beta' - \cos\beta\cos\alpha' = 0,$$

whence $\cos\alpha = \cos\alpha'$, $\cos\beta = \cos\beta'$, and $\cos\gamma = \cos\gamma'$ (as is evident from the definition of direction-cosines); or

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}.$$

Ex. 1. If P, Q are $(2, 3, -6)$, $(3, -4, 5)$, find the angle that OP makes with OQ.

Ans. $\cos\theta = \dfrac{-18\sqrt{2}}{35}$.

Ex. 2. P, Q, R are $(2, 3, 5)$, $(-1, 3, 2)$, $(3, 5, -2)$. Find the angles of the triangle PQR.

Ans. 90°, $\cos^{-1}\dfrac{\sqrt{6}}{3}$, $\cos^{-1}\dfrac{\sqrt{3}}{3}$.

Ex. 3. Find the angles between the lines whose direction-cosines are proportional to (i) 2, 3, 4 ; 3, 4, 5 ; (ii) 2, 3, 4 ; 1, -2, 1.

Ans. (i) $\cos^{-1}\dfrac{38}{5\sqrt{58}}$, (ii) 90°.

Ex. 4. The lines whose direction-cosines are proportional to 2, 1, 1 ; 4, $\sqrt{3}-1$, $-\sqrt{3}-1$; 4, $-\sqrt{3}-1$, $\sqrt{3}-1$ are inclined to one another at an angle $\pi/3$.

Ex. 5. If l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 are the direction-cosines of three mutually perpendicular lines, the line whose direction-cosines are proportional to $l_1+l_2+l_3$, $m_1+m_2+m_3$, $n_1+n_2+n_3$ makes equal angles with them.

Ex. 6. Find the angle between two diagonals of a cube.

Ans. $\cos^{-1} 1/3$.

Ex. 7. Prove by direction-cosines that the points $(3, 2, 4)$, $(4, 5, 2)$, $(5, 8, 0)$, $(2, -1, 6)$ are collinear.

Ex. 8. A line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube; prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = 4/3.$$

Ex. 9. If the edges of a rectangular parallelepiped are a, b, c, shew that the angles between the four diagonals are given by

$$\cos^{-1}\left(\frac{a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right).$$

Ex. 10. If a variable line in two adjacent positions has direction-cosines l, m, n ; $l+\delta l$, $m+\delta m$, $n+\delta n$, shew that the small angle, $\delta\theta$, between the two positions is given by $\delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2$.

We have $\Sigma l^2 = 1$ and $\Sigma(l+\delta l)^2 = 1$, therefore $\Sigma(\delta l)^2 = -2\Sigma l\,\delta l$.

But
$$\cos\delta\theta = \Sigma l(l+\delta l) = 1 + \Sigma l\,\delta l.$$

Therefore
$$2\sin^2\frac{\delta\theta}{2} = -\Sigma l\,\delta l = \frac{1}{2}\Sigma(\delta l)^2.$$

That is, since
$$\sin\frac{\delta\theta}{2} = \frac{\delta\theta}{2},\quad \delta\theta^2 = \Sigma(\delta l)^2.$$

Ex. 11. Lines OA, OB are drawn from O with direction-cosines proportional to $(1, -2, -1)$, $(3, -2, 3)$. Find the direction-cosines of the normal to the plane AOB.

Ans. $\dfrac{4}{\sqrt{29}}, \dfrac{3}{\sqrt{29}}, \dfrac{-2}{\sqrt{29}}$.

Ex. 12. Prove that the three lines drawn from O with direction-cosines proportional to $(1, -1, 1)$, $(2, -3, 0)$, $(1, 0, 3)$ lie in one plane.

Ex. 13. Prove that the three lines drawn from O with direction-cosines l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3, are coplanar if

$$\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} = 0.$$

Ex. 14. Find the direction-cosines of the axis of the right circular cone which passes through the lines drawn from O with direction-cosines proportional to $(3, 6, -2)$, $(2, 2, -1)$, $(-1, 2, 2)$, and prove that the cone also passes through the coordinate axes.

Ans. $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$.

Ex. 15. Lines are drawn from O with direction-cosines proportional to $(1, 2, 2)$, $(2, 3, 6)$, $(3, 4, 12)$. Prove that the axis of the right circular cone through them has direction-cosines $-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$, and that the semivertical angle of the cone is $\cos^{-1} 1/\sqrt{3}$.

24. Distance of a point from a line. *To find the distance of* P, (x', y', z') *from the line through* A, (a, b, c), *whose direction-cosines are* $\cos\alpha, \cos\beta, \cos\gamma$.

Let PN, the perpendicular from P to the line, have measure δ. Then AN is the projection of AP on the line, and its measure is, (Ex. 3, § 21),

$$(x'-a)\cos\alpha + (y'-b)\cos\beta + (z'-c)\cos\gamma.$$

But
$$PN^2 = AP^2 - AN^2,$$
therefore

$$\begin{aligned}\delta^2 = \{(x'-a)^2 + (y'-b)^2 + (z'-c)^2\}(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) \\ - \{(x'-a)\cos\alpha + (y'-b)\cos\beta + (z'-c)\cos\gamma\}^2,\end{aligned}$$

which, by Lagrange's identity, gives

$$\begin{aligned}\delta^2 = &\{(y'-b)\cos\gamma - (z'-c)\cos\beta\}^2 \\ &+ \{(z'-c)\cos\alpha - (x'-a)\cos\gamma\}^2 \\ &+ \{(x'-a)\cos\beta - (y'-b)\cos\alpha\}^2.\end{aligned}$$

Cor. If (x', y', z') is any point on the line, $\delta = 0$, and

$$\frac{x'-a}{\cos\alpha} = \frac{y'-b}{\cos\beta} = \frac{z'-c}{\cos\gamma}. \quad \text{(Cf. § 21, Cor. 3.)}$$

Ex. 1. Find the distance of $(-1, 2, 5)$ from the line through $(3, 4, 5)$ whose direction-cosines are proportional to $2, -3, 6$.

Ans. $\dfrac{4\sqrt{61}}{7}$.

Ex. 2. Find the distance of A, $(1, -2, 3)$ from the line, PQ, through P, $(2, -3, 5)$, which makes equal angles with the axes.

Ans. $\sqrt{\dfrac{14}{3}}$.

Ex. 3. Shew that the equation to the right circular cone whose vertex is at the origin, whose axis has direction-cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$, and whose semivertical angle is θ, is

$$(y\cos\gamma - z\cos\beta)^2 + (z\cos\alpha - x\cos\gamma)^2 + (x\cos\beta - y\cos\alpha)^2 = \sin^2\theta(x^2+y^2+z^2).$$

Ex. 4. Find the equation to the right circular cone whose vertex is P, axis PQ (Ex. 2), and semivertical angle is 30°.

Ans. $4\{(y-z+8)^2+(z-x-3)^2+(x-y-5)^2\} = 3\{(x-2)^2+(y+3)^2+(z-5)^2\}$.

Ex. 5. Find the equation to the right circular cone whose vertex is P, axis PQ, and which passes through A (Ex. 2).

Ans. $3\{(y-z+8)^2+(z-x-3)^2+(x-y-5)^2\} = 7\{(x-2)^2+(y+3)^2+(z-5)^2\}$.

Ex. 6. The axis of a right cone, vertex O, makes equal angles with the coordinate axes, and the cone passes through the line drawn from O with direction-cosines proportional to $(1, -2, 2)$. Find the equation to the cone. *Ans.* $4x^2+4y^2+4z^2+9yz+9zx+9xy=0$.

Ex. 7. Find the equation to the right circular cylinder of radius 2 whose axis passes through $(1, 2, 3)$ and has direction-cosines proportional to $(2, -3, 6)$.

Ans. $9(2y+z-7)^2+4(z-3x)^2+(3x+2y-7)^2=196$.

***25. Direction-cosines referred to oblique axes.** Let X′OX, Y′OY, Z′OZ, (fig. 20), be oblique axes, the angles YOZ, ZOX, XOY being λ, μ, ν respectively. Let A′OA be the line through O whose direction-cosines are $\cos\alpha$, $\cos\beta$, $\cos\gamma$. Take P, (x, y, z) any point on A′OA, and let the measure of OP be r. Draw PN parallel to OZ to meet the plane XOY in N, and NM parallel to OY to meet OX in M. Then, since the projection of OP is equal to the

sum of the projections of OM, MN, NP, projecting on OX, OY, OZ, OP in turn, we obtain

$$r\cos\alpha = x + y\cos\nu + z\cos\mu, \quad \ldots\ldots\ldots\ldots(1)$$

$$r\cos\beta = x\cos\nu + y + z\cos\lambda, \quad \ldots\ldots\ldots\ldots(2)$$

$$r\cos\gamma = x\cos\mu + y\cos\lambda + z, \quad \ldots\ldots\ldots\ldots(3)$$

$$r = x\cos\alpha + y\cos\beta + z\cos\gamma. \quad \ldots\ldots\ldots\ldots(4)$$

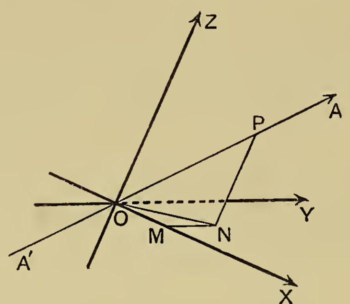

FIG. 20.

Therefore, eliminating r, x, y, z, we have the relation satisfied by the direction-cosines of any line

$$\begin{vmatrix} 1, & \cos\nu, & \cos\mu, & \cos\alpha \\ \cos\nu, & 1, & \cos\lambda, & \cos\beta \\ \cos\mu, & \cos\lambda, & 1, & \cos\gamma \\ \cos\alpha, & \cos\beta, & \cos\gamma, & 1 \end{vmatrix} = 0,$$

which may be written,

$$\Sigma\sin^2\lambda\cos^2\alpha - 2\Sigma(\cos\lambda - \cos\mu\cos\nu)\cos\beta\cos\gamma$$
$$= 1 - \cos^2\lambda - \cos^2\mu - \cos^2\nu + 2\cos\lambda\cos\mu\cos\nu.$$

Cor. 1. Multiply (1), (2), (3) by x, y, z respectively, and add, then

$$x^2 + y^2 + z^2 + 2yz\cos\lambda + 2zx\cos\mu + 2xy\cos\nu$$
$$= r(x\cos\alpha + y\cos\beta + z\cos\gamma),$$
$$= r^2, \text{ [by (4)]}. \quad \ldots\ldots\ldots\ldots(\text{A})$$

Cor. 2. If P, Q are (x_1, y_1, z_1), (x_2, y_2, z_2), PQ^2 is given by

$$\Sigma(x_2 - x_1)^2 + 2\Sigma(y_2 - y_1)(z_2 - z_1)\cos\lambda.$$

Ex. 1. If P, (x, y, z) is any point on the plane through O at right angles to OX, the projection of OP on OX is zero, and therefore

$$x+y\cos\nu+z\cos\mu=0.$$

Ex. 2. If P, (x, y, z) is any point on the normal through O to the plane XOY, $x+y\cos\nu+z\cos\mu=0=x\cos\nu+y+z\cos\lambda$.

***26.** If a, b, c are given proportionals to the direction-cosines of a line, the actual direction-cosines are given by

$$\frac{\cos\alpha}{a}=\frac{\cos\beta}{b}=\frac{\cos\gamma}{c}$$

$$=\frac{\pm\{\Sigma\sin^2\lambda\cos^2\alpha-2\Sigma(\cos\lambda-\cos\mu\cos\nu)\cos\beta\cos\gamma)^{\frac{1}{2}}}{\{\Sigma\sin^2\lambda\,.\,a^2-2\Sigma(\cos\lambda-\cos\mu\cos\nu)bc\}^{\frac{1}{2}}}$$

$$=\frac{\pm\{1-\cos^2\lambda-\cos^2\mu-\cos^2\nu+2\cos\lambda\cos\mu\cos\nu\}^{\frac{1}{2}}}{\{\Sigma\sin^2\lambda\,.\,a^2-2\Sigma(\cos\lambda-\cos\mu\cos\nu)bc\}^{\frac{1}{2}}}.$$

***27. The angle between two lines.** If OQ has direction-cosines $\cos\alpha'$, $\cos\beta'$, $\cos\gamma'$, and makes an angle θ with OP, projecting on OQ, we obtain

$$r\cos\theta=x\cos\alpha'+y\cos\beta'+z\cos\gamma'. \quad\ldots\ldots\ldots\ldots(5)$$

Therefore eliminating x, y, z, r between equations (1), (2), (3) of § 25, and (5), we have

$$\begin{vmatrix} 1, & \cos\nu, & \cos\mu, & \cos\alpha \\ \cos\nu, & 1, & \cos\lambda, & \cos\beta \\ \cos\mu, & \cos\lambda, & 1, & \cos\gamma \\ \cos\alpha', & \cos\beta', & \cos\gamma', & \cos\theta \end{vmatrix}=0, \quad \text{or}$$

$$\begin{aligned}\Sigma(\sin^2\lambda\cos\alpha\cos\alpha')-\Sigma\{(\cos\lambda-\cos\mu\cos\nu) \\ \times(\cos\beta\cos\gamma'+\cos\beta'\cos\gamma)\} \\ =\cos\theta(1-\cos^2\lambda-\cos^2\mu-\cos^2\nu+2\cos\lambda\cos\mu\cos\nu).\end{aligned}$$

Cor. The angles between the lines whose direction-cosines are proportional to $a, b, c : a', b', c'$ are given by

$$\cos\theta=\frac{\pm\{\Sigma(aa'\sin^2\lambda)-\Sigma(bc'+b'c)(\cos\lambda-\cos\mu\cos\nu)\}}{\{\Sigma a^2\sin^2\lambda-2\Sigma bc(\cos\lambda-\cos\mu\cos\nu)\}^{\frac{1}{2}}\times\{\Sigma a'^2\sin^2\lambda-2\Sigma b'c'(\cos\lambda-\cos\mu\cos\nu)\}^{\frac{1}{2}}}.$$

Ex. 1. If $\lambda=\mu=\nu=\pi/3$, find the angles between the lines whose direction-cosines are proportional to

(i) $2, 3, 4$; $3, 4, 5$; (ii) $2, 3, 4$: $1, -2, 1$.

Ans. (i) $\cos^{-1}\dfrac{22}{7\sqrt{10}}$; (ii) $\pi/2$.

Ex. 2. Prove that the lines whose direction-cosines are proportional to l, m, n; $m-n, n-l, l-m$ are at right angles if $\lambda=\mu=\nu$.

Ex. 3. The edges OA, OB, OC of a tetrahedron are of lengths a, b, c, and the angles BOC, COA, AOB are λ, μ, ν; find the volume.

Take OA, OB, OC as axes, and draw CN at right angles to the plane AOB. Then if CN is of length p, and V denotes the volume, $V=\frac{1}{3}\cdot\frac{ab\sin\nu}{2}\cdot p$, and $p=c\cos\angle OCN$. But the direction-cosines of CN are 0, 0, $\cos\angle OCN$, therefore, by § 25,

$$\sin^2\nu\cos^2\angle OCN=1-\cos^2\lambda-\cos^2\mu-\cos^2\nu+2\cos\lambda\cos\mu\cos\nu,$$

$$\therefore\ V=\frac{abc}{6}\{1-\cos^2\lambda-\cos^2\mu-\cos^2\nu+2\cos\lambda\cos\mu\cos\nu\}^{\frac{1}{2}}.$$

DIRECTION-RATIOS.

28. Let OL be drawn from O in the same direction as a given directed line PQ and of unit length. Then the coordinates of L evidently depend only on the direction of PQ, and when given, determine that direction. They are therefore called the **direction-ratios** of PQ.

If the axes are rectangular the direction-ratios are the same as the direction-cosines.

29. *If* P, (x, y, z) *is any point on a given line* A′OA *whose direction-ratios are* l, m, n, *and the measure of* OP *is* r, *then*

$$l=\frac{x}{r},\quad m=\frac{y}{r},\quad n=\frac{z}{r}.$$

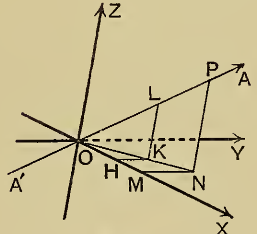

FIG. 21.

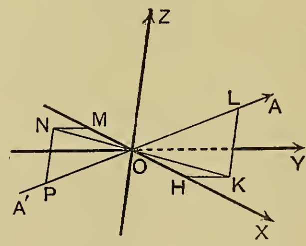

FIG. 22.

In fig. 21, r is positive, in fig. 22 r is negative. LK, PN are parallel to OZ; KH, NM are parallel to OY. Then since the parallel planes PNM, LKH cut X′OX, A′OA proportionally,

$$OP : OL = OM : OH,$$

where OP, OL, OM, OH are directed segments.

But the measures of OM and OH are x and l respectively, and therefore $l = x/r$. Similarly, $m = y/r$, $n = z/r$.

Cor. 1. If P, (x, y, z) is any point on the line through O whose direction-ratios are l, m, n,

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}. \quad \text{(Cf. § 21, Cor. 2.)}$$

Cor. 2. If (x, y, z) is any point on the line through (x', y', z') whose direction-ratios are l, m, n,

$$\frac{x-x'}{l} = \frac{y-y'}{m} = \frac{z-z'}{n}. \quad \text{(Cf. § 21, Cor. 3.)}$$

Cor. 3. If P, Q are (x_1, y_1, z_1), (x_2, y_2, z_2), and the measure of PQ is r, the direction-ratios of PQ are

$$\frac{x_2-x_1}{r}, \quad \frac{y_2-y_1}{r}, \quad \frac{z_2-z_1}{r}.$$

Ex. 1. Find the direction-ratios of the lines bisecting the angles between the lines whose direction-ratios are l_1, m_1, n_1; l_2, m_2, n_2.

If L, L′ are (l_1, m_1, n_1), (l_2, m_2, n_2), then OL and OL′ are the lines from O with the given direction-ratios, and OL and OL′ are of unit length.

The mid-point, M, of LL′ has coordinates, $\frac{l_1+l_2}{2}, \frac{m_1+m_2}{2}, \frac{n_1+n_2}{2}$, and $\mathrm{OM} = \cos\frac{\theta}{2}$, where $\angle \mathrm{LOL}' = \theta$, therefore the direction-ratios of OM are

$$\frac{l_1+l_2}{2\cos\theta/2}, \quad \frac{m_1+m_2}{2\cos\theta/2}, \quad \frac{n_1+n_2}{2\cos\theta/2}.$$

Similarly, the direction-ratios of the other bisector are $\frac{l_1-l_2}{2\sin\theta/2}$, etc.

Ex. 2. OX, OY, OZ are given rectangular axes;

$\mathrm{OX_1}, \mathrm{OY_1}, \mathrm{OZ_1}$ bisect the angles YOZ, ZOX, XOY;

$\mathrm{OX_2}, \mathrm{OY_2}, \mathrm{OZ_2}$ bisect the angles $\mathrm{Y_1OZ_1}, \mathrm{Z_1OX_1}, \mathrm{X_1OY_1}$.

Prove that $\angle \mathrm{Y_1OZ_1} = \angle \mathrm{Z_1OX_1} = \angle \mathrm{X_1OY_1} = \pi/3$, and that

$$\angle \mathrm{Y_2OZ_2} = \angle \mathrm{Z_2OX_2} = \angle \mathrm{X_2OY_2} = \cos^{-1} 5/6.$$

Ex. 3. A, B, C, are the points $(1, 2, 3)$, $(3, 5, -3)$, $(-2, 6, 15)$, and the axes are rectangular. Find the direction-cosines of the interior bisector of the angle BAC. *Ans.* $1/\sqrt{182}$, $67/5\sqrt{182}$, $6/5\sqrt{182}$.

***30.** The direction-ratios of any line satisfy the equation (§ 25, Cor. 1, (A)),

$$l^2 + m^2 + n^2 + 2mn\cos\lambda + 2nl\cos\mu + 2lm\cos\nu = 1,$$

which it is convenient to write, $\phi(l, m, n) = 1$.

*31. *To find the direction-cosines of the line whose direction-ratios are l, m, n.*

Project OL, (figs. 21 and 22), on the axes and on itself, and we obtain, as in § 25 (1), (2), (3), (4),

$$\cos\alpha = l + m\cos\nu + n\cos\mu \equiv \frac{1}{2}\frac{\partial\phi}{\partial l},$$

$$\cos\beta = l\cos\nu + m + n\cos\lambda \equiv \frac{1}{2}\frac{\partial\phi}{\partial m},$$

$$\cos\gamma = l\cos\mu + m\cos\lambda + n \equiv \frac{1}{2}\frac{\partial\phi}{\partial n},$$

$$1 = l\cos\alpha + m\cos\beta + n\cos\gamma. \quad \text{(Cf. § 21 (3).)}$$

*32. *To find the angles between the lines whose direction-ratios are l, m, n; l', m', n'.*

Let OL′, the unit ray from O which has direction-ratios l', m', n', make an angle θ with OL. Then projecting OL′ on OL, we obtain,

$$\begin{aligned}\cos\theta &= l'\cos\alpha + m'\cos\beta + n'\cos\gamma,\\ &= \frac{1}{2}\left(l'\frac{\partial\phi}{\partial l} + m'\frac{\partial\phi}{\partial m} + n'\frac{\partial\phi}{\partial n}\right), \quad \text{(by § 31)},\\ &= ll' + mm' + nn' + (mn' + m'n)\cos\lambda\\ &\qquad + (nl' + n'l)\cos\mu + (lm' + l'm)\cos\nu,\\ &= \frac{1}{2}\left(l\frac{\partial\phi}{\partial l'} + m\frac{\partial\phi}{\partial m'} + n\frac{\partial\phi}{\partial n'}\right).\end{aligned}$$

Cor. If the lines are at right angles,

$$l\frac{\partial\phi}{\partial l'} + m\frac{\partial\phi}{\partial m'} + n\frac{\partial\phi}{\partial n'} = 0,$$

which may be written in the forms,

$$l'\cos\alpha + m'\cos\beta + n'\cos\gamma = 0$$

$$\text{or} \quad l\cos\alpha' + m\cos\beta' + n\cos\gamma' = 0,$$

where $\cos\alpha'$, $\cos\beta'$, $\cos\gamma'$ are the direction-cosines of OL′.

Ex. 1. If $\lambda = \mu = \nu = \pi/3$, find the direction-ratios of the line joining the origin to the point $(1, 2, -1)$. Find also the direction-cosines.

Ans. $\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}; \ \frac{3}{2\sqrt{5}}, \frac{2}{\sqrt{5}}, \frac{5}{2\sqrt{5}}.$

Ex. 2. Shew that the direction-ratios of a normal to the plane **XOY** are given by

$$\frac{l}{\cos\nu\cos\lambda-\cos\mu}=\frac{m}{\cos\mu\cos\nu-\cos\lambda}=\frac{n}{\sin^2\nu}=\frac{1}{\sin\nu\Delta^{\frac{1}{2}}},$$

where $\Delta\equiv 1-\cos^2\lambda-\cos^2\mu-\cos^2\nu+2\cos\lambda\cos\mu\cos\nu.$

Ex. 3. Prove that the lines which bisect the angles **YOZ, ZOX, XOY,** internally, have direction-cosines

$$\frac{\cos\mu+\cos\nu}{2\cos\lambda/2},\quad \cos\frac{\lambda}{2},\quad \cos\frac{\lambda}{2};\ \text{etc.,}$$

and that the angles between them are

$$\cos^{-1}\left(\frac{1+\cos\lambda+\cos\mu+\cos\nu}{4\cos\mu/2\cos\nu/2}\right),\ \text{etc}$$

CHAPTER III

THE PLANE.

33. Let ABC, (fig. 23), a given plane, make intercepts OA, OB, OC on the axes, measured by a, b, c; and let ON, the normal from O to the plane, have direction-cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$, and have measure p, (p is a positive number).

Equation to a plane. (i) *To find the equation to the plane* ABC *in terms of* $\cos\alpha$, $\cos\beta$, $\cos\gamma$, p.

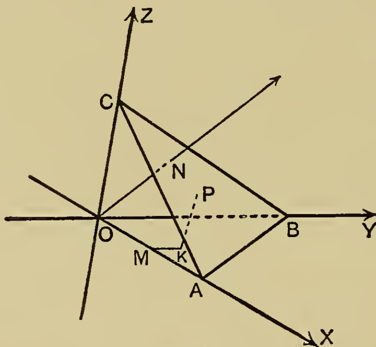

FIG. 23.

Let P, (x, y, z) be any point on the plane. Draw PK parallel to OZ to meet the plane XOY in K, and KM parallel to OY to meet OX in M. Then the measures of OM, MK, KP are x, y, z respectively, and since ON is the projection of OP on ON, and therefore equal to the sum of the projections of OM, MK, KP on ON,

$$p = x\cos\alpha + y\cos\beta + z\cos\gamma.$$

This equation, satisfied by the coordinates of every point on the plane, represents the plane.

(ii) *To find the equation to the plane in terms of a, b, c.*

$$\mathrm{ON} = \text{projection of OA on ON} = \mathrm{OA}\cos\alpha;$$

$$\therefore\ p = a\cos\alpha. \quad \text{Similarly, } b\cos\beta = c\cos\gamma = p.$$

Hence, by (i), the equation to the plane is

$$\frac{x\cos\alpha}{p} + \frac{y\cos\beta}{p} + \frac{z\cos\gamma}{p} = 1,$$

$$\text{i.e.}\quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

Ex. Find the intercepts made on the coordinate axes by the plane $x+2y-2z=9$. Find also the direction-cosines of the normal to the plane if the axes are rectangular. *Ans.* 9, 9/2, $-9/2$; $\frac{1}{3}$, $\frac{2}{3}$, $-\frac{2}{3}$.

34. General equation to a plane. *The general equation of the first degree in x, y, z represents a plane.*

For $\mathrm{A}x + \mathrm{B}y + \mathrm{C}z + \mathrm{D} = 0$ can be written

$$\frac{x}{-\mathrm{D}/\mathrm{A}} + \frac{y}{-\mathrm{D}/\mathrm{B}} + \frac{z}{-\mathrm{D}/\mathrm{C}} = 1,$$

and therefore represents a plane making intercepts $-\mathrm{D}/\mathrm{A}$, $-\mathrm{D}/\mathrm{B}$, $-\mathrm{D}/\mathrm{C}$ on the axes.

35. If $\mathrm{A}x + \mathrm{B}y + \mathrm{C}z + \mathrm{D} = 0$ and $p = x\cos\alpha + y\cos\beta + z\cos\gamma$ represent the same plane,

$$\frac{\cos\alpha}{-\mathrm{A}} = \frac{\cos\beta}{-\mathrm{B}} = \frac{\cos\gamma}{-\mathrm{C}} = \frac{p}{\mathrm{D}}; \quad\text{.................. (1)}$$

therefore the direction-cosines of the normal to the plane $\mathrm{A}x + \mathrm{B}y + \mathrm{C}z + \mathrm{D} = 0$ are proportional to A, B, C. If the axes are rectangular, each of the ratios in (1) $= \dfrac{\pm 1}{\sqrt{\mathrm{A}^2+\mathrm{B}^2+\mathrm{C}^2}}$.

But p is a positive number; therefore if D is positive,

$$p = \frac{\mathrm{D}}{\sqrt{\mathrm{A}^2+\mathrm{B}^2+\mathrm{C}^2}}, \quad \cos\alpha = \frac{-\mathrm{A}}{\sqrt{\mathrm{A}^2+\mathrm{B}^2+\mathrm{C}^2}},$$

$$\cos\beta = \frac{-\mathrm{B}}{\sqrt{\mathrm{A}^2+\mathrm{B}^2+\mathrm{C}^2}}, \quad\text{and}\quad \cos\gamma = \frac{-\mathrm{C}}{\sqrt{\mathrm{A}^2+\mathrm{B}^2+\mathrm{C}^2}}.$$

If D is negative, we must change the sign of $\sqrt{\mathrm{A}^2+\mathrm{B}^2+\mathrm{C}^2}$.

Cor. If the axes are rectangular, the angle between the planes $ax+by+cz+d=0,\quad a'x+b'y+c'z+d'=0$

$$\text{is}\quad \cos^{-1}\left\{\frac{\pm(aa'+bb'+cc')}{\sqrt{a^2+b^2+c^2}\sqrt{a'^2+b'^2+c'^2}}\right\}.$$

Ex. 1. If the axes are rectangular, find the angle between the planes

(i) $2x-y+z=6,\quad x+y+2z=3$;

(ii) $3x+4y-5z=9,\quad 2x+6y+6z=7$.

Ans. (i) $\pi/3$, (ii) $\pi/2$.

Ex. 2. If the axes are rectangular, find the distance of the origin from the plane $6x-3y+2z-14=0$. *Ans.* 2.

Ex. 3. Shew that the equations $by+cz+d=0$, $cz+ax+d=0$, $ax+by+d=0$ represent planes parallel to OX, OY, OZ respectively. Find the equations to the planes through the points (2, 3, 1), (4, −5, 3) parallel to the coordinate axes.

Ans. $y+4z-7=0,\quad x-z-1=0,\quad 4x+y-11=0$.

Ex. 4. Find the equation to the plane through (1, 2, 3) parallel to $3x+4y-5z=0$. *Ans.* $3x+4y-5z+4=0$.

Ex. 5. Prove that the equation to the plane through (α, β, γ) parallel to $ax+by+cz=0$ is $ax+by+cz=a\alpha+b\beta+c\gamma$.

Ex. 6. If the axes are rectangular and P is the point (2, 3, −1), find the equation to the plane through P at right angles to OP.

Ans. $2x+3y-z=14$.

Ex. 7. Prove that the equation $2x^2-6y^2-12z^2+18yz+2zx+xy=0$ represents a pair of planes, and find the angle between them.

Ans. $\cos^{-1} 16/21$.

Ex. 8. Prove that the equation

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

represents a pair of planes if $abc+2fgh-af^2-bg^2-ch^2=0$.

Prove that the angle between the planes is

$$\tan^{-1}\left(\frac{2(f^2+g^2+h^2-bc-ca-ab)^{\frac{1}{2}}}{a+b+c}\right).$$

Ex. 9. A variable plane is at a constant distance p from the origin and meets the axes, which are rectangular, in A, B, C. Through A, B, C planes are drawn parallel to the coordinate planes. Shew that the locus of their point of intersection is given by $x^{-2}+y^{-2}+z^{-2}=p^{-2}$.

36. Plane through three given points. The general equation to a plane contains three arbitrary constants, and therefore a plane can be found to satisfy three conditions which each involve one relation between the constants;

e.g. a plane can be found to pass through any three non-collinear points.

To find the equation to the plane through (x_1, y_1, z_1) (x_2, y_2, z_2), (x_3, y_3, z_3).

Let the equation to the plane be $ax+by+cz+d=0$.

Then
$$ax_1+by_1+cz_1+d=0,$$
$$ax_2+by_2+cz_2+d=0,$$
$$ax_3+by_3+cz_3+d=0.$$

Therefore, eliminating a, b, c, d, we obtain the required equation,
$$\begin{vmatrix} x, & y, & z, & 1 \\ x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \end{vmatrix}=0.$$

Ex. 1. Find the equation to the plane through the three points
$$(1, 1, 0), \quad (1, 2, 1), \quad (-2, 2, -1).$$
Ans. $2x+3y-3z=5$.

Ex. 2. Shew that the four points $(0, -1, 0)$, $(2, 1, -1)$, $(1, 1, 1)$, $(3, 3, 0)$ are coplanar.

37. Distance from a point to a plane. *To find the distance of the point* P, (x', y', z') *from the plane*
$$p=x\cos\alpha+y\cos\beta+z\cos\gamma.$$

Suppose that p is a positive number so that $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction-cosines of the normal *from* the origin *to* the plane. Change the origin to (x', y', z'), and the equation to the plane becomes
$$p=(x+x')\cos\alpha+(y+y')\cos\beta+(z+z')\cos\gamma,$$
or
$$p'=x\cos\alpha+y\cos\beta+z\cos\gamma,$$
where
$$p'\equiv p-x'\cos\alpha-y'\cos\beta-z'\cos\gamma.$$

Hence the distance of (x', y', z'), the new origin, from the plane is
$$p'\equiv p-x'\cos\alpha-y'\cos\beta-z'\cos\gamma.$$

If P is on the same side of the plane as the original origin O, $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are still the direction-cosines of the normal *from* the new origin, P, *to* the plane, and therefore p' or $p-x'\cos\alpha-y'\cos\beta-z'\cos\gamma$ is positive. If P

and O are on opposite sides of the plane, $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction-cosines of the normal *from* the plane *to* P, and therefore p' or $p-x'\cos\alpha-y'\cos\beta-z'\cos\gamma$ is negative. Hence, if p is positive,

$$p-x'\cos\alpha-y'\cos\beta-z'\cos\gamma$$

is positive if (x', y', z') is any point on the same side of the plane as the origin, and negative if (x', y', z') is any point on the side of the plane remote from the origin.

Cor. 1. The distance of (x', y', z') from the plane

$$ax+by+cz+d=0,$$

if the axes are rectangular, is given by $\dfrac{ax'+by'+cz'+d}{\pm\sqrt{a^2+b^2+c^2}}$.

If d is positive the positive sign is to be taken, as it gives a positive value for the perpendicular from the origin.

Cor. 2. If d is positive, the expression $ax'+by'+cz'+d$ is positive if (x', y', z') and the origin are on the same side of the plane $ax+by+cz+d=0$, and negative if they are on opposite sides.

Ex. 1. If P is (x', y', z'), shew that the projection of OP on the normal to the plane

$$p=x\cos\alpha+y\cos\beta+z\cos\gamma \text{ is } x'\cos\alpha+y'\cos\beta+z'\cos\gamma,$$

and deduce the results of § 37.

Ex. 2. Find the distances of the points $(2, 3, -5)$, $(3, 4, 7)$ from the plane $x+2y-2z=9$. Are the points on the same side of the plane?

Ans. 3, 4, No.

Ex. 3. Find the locus of a point whose distance from the origin is 7 times its distance from the plane $2x+3y-6z=2$.

Ans. $3x^2+8y^2+35z^2-36yz-24zx+12xy-8x-12y+24z+4=0$.

Ex. 4. Find the locus of a point the sum of the squares of whose distances from the planes $x+y+z=0$, $x-z=0$, $x-2y+z=0$, is 9.

Ans. $x^2+y^2+z^2=9$.

Ex. 5. The sum of the squares of the distances of a point from the planes $x+y+z=0$, $x-2y+z=0$ is equal to the square of its distance from the plane $x=z$. Prove that the equation to the locus of the point is $y^2+2xz=0$. By turning the axes of x and z in their plane through angles of 45°, prove that the locus is a right circular cone whose semi-vertical angle is 45°.

38. Planes bisecting the angles between given planes. *To find the planes bisecting the angles between the given planes* $ax+by+cz+d=0$, $a'x+b'y+c'z+d'=0$, *the axes being rectangular.*

We can always write the equations so that d and d' are positive. Then the equation

$$\frac{ax+by+cz+d}{\sqrt{a^2+b^2+c^2}}=\frac{a'x+b'y+c'z+d'}{\sqrt{a'^2+b'^2+c'^2}}$$

represents the locus of points equidistant from the given planes, and since the expressions

$$ax+by+cz+d, \quad a'x+b'y+c'z+d'$$

in the equation have the same sign, the points are on the origin side of both planes or on the non-origin side of both. The locus is therefore the plane bisecting that angle between the given planes which contains the origin. Similarly,

$$\frac{ax+by+cz+d}{\sqrt{a^2+b^2+c^2}}=-\frac{a'x+b'y+c'z+d'}{\sqrt{a'^2+b'^2+c'^2}}$$

represents the plane bisecting the other angle between the given planes.

Ex. 1. Shew that the origin lies in the acute angle between the planes $x+2y+2z=9$, $4x-3y+12z+13=0$. Find the planes bisecting the angles between them, and point out which bisects the acute angle.

Ans. Acute, $25x+17y+62z-78=0$; obtuse, $x+35y-10z-156=0$.

***Ex. 2.** Shew that the plane $ax+by+cz+d=0$ divides the join of (x_1, y_1, z_1), (x_2, y_2, z_2) in the ratio

$$-\frac{ax_1+by_1+cz_1+d}{ax_2+by_2+cz_2+d}.$$

[The point $\left(\frac{\lambda x_2+x_1}{\lambda+1}, \frac{\lambda y_2+y_1}{\lambda+1}, \frac{\lambda z_2+z_1}{\lambda+1}\right)$ lies on the plane if

$$\lambda(ax_2+by_2+cz_2+d)+ax_1+by_1+cz_1+d=0.]$$

***Ex. 3.** Hence shew that the planes $u\equiv ax+by+cz+d=0$, $v\equiv a'x+b'y+c'z+d'=0$, $u+\lambda v=0$, $u-\lambda v=0$ divide any transversal harmonically.

Let P, (x_1, y_1, z_1) be on the plane $u=0$, then $u_1\equiv ax_1+by_1+cz_1+d=0$. Let Q, (x_2, y_2, z_2) be on the plane $v=0$, then $v_2\equiv a'x_2+b'y_2+c'z_2+d'=0$. The planes $u\pm\lambda v=0$ divide PQ in the ratios

$$-\frac{u_1\pm\lambda v_1}{u_2\pm\lambda v_2}, \quad \text{i.e. } \mp\frac{\lambda v_1}{u_2},$$

i.e. divide PQ harmonically.

*Ex. 4. If A, P, B, Q are any four collinear points, the anharmonic ratio, or cross-ratio of the range APBQ, is defined to be

$$\frac{AP}{PB} \div \frac{AQ}{QB} \text{ or } \frac{AP.QB}{AQ.PB}.$$

Prove that four given planes that pass through one line cut any transversal in a range of constant cross-ratio.

If $u=0$, $v=0$ are two planes through the line, the equations to the four given planes can be written, $u+\lambda_r v=0$, $r=1, 2, 3, 4$. Let A, B, (x_1, y_1, z_1), (x_2, y_2, z_2) lie on the planes $u+\lambda_1 v=0$, $u+\lambda_3 v=0$ respectively. Then $u_1+\lambda_1 v_1=0$ and $u_2+\lambda_3 v_2=0$. If P, Q lie on the planes $u+\lambda_2 v=0$, $u+\lambda_4 v=0$, then by Ex. 2,

$$\frac{AP}{PB}=-\frac{u_1+\lambda_2 v_1}{u_2+\lambda_2 v_2},\quad \frac{AQ}{QB}=-\frac{u_1+\lambda_4 v_1}{u_2+\lambda_4 v_2},$$

and therefore

$$\frac{AP.QB}{AQ.PB}=\frac{(\lambda_1-\lambda_2)(\lambda_3-\lambda_4)}{(\lambda_3-\lambda_2)(\lambda_1-\lambda_4)}.$$

This constant cross-ratio is called the cross-ratio of the four planes.

*Ex. 5. P, Q, R, S are four coplanar points on the sides AB, BC, CD, DA of a skew quadrilateral. Prove that

$$\frac{AP}{PB}\cdot\frac{BQ}{QC}\cdot\frac{CR}{RD}\cdot\frac{DS}{SA}=1.$$

THE STRAIGHT LINE.

39. The equations to a line. Every equation of the first degree represents a plane. Two equations of the first degree are satisfied by the coordinates of any point on the line of intersection of the planes which they represent, and therefore the two equations together represent that line. Thus $ax+by+cz+d=0$, $a'x+b'y+c'z+d'=0$ represent a straight line.

40. Symmetrical form of equations. The equations to a straight line can be found in a more symmetrical form. If the line passes through a given point P, (x', y', z') and has direction-ratios l, m, n,

$$l=\frac{x-x'}{r},\quad m=\frac{y-y'}{r},\quad n=\frac{z-z'}{r},$$

where Q, (x, y, z) is any point on it, and the measure of PQ is r, (§ 21, Cor. 3; § 29, Cor. 3). And therefore the

coordinates of any point on the line satisfy the equations

$$\frac{x-x'}{l}=\frac{y-y'}{m}=\frac{z-z'}{n} \quad (=r).$$

These equations enable us to express the coordinates of a variable point on the line in terms of one parameter r, for

$$x=x'+lr, \quad y=y'+mr, \quad z=z'+nr.$$

Conversely, any equations of the form

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}$$

represent a straight line passing through the point (a, b, c) and having direction-ratios proportional to l, m, n.

Ex. 1. Find where the line $\dfrac{x-1}{2}=\dfrac{y-2}{-3}=\dfrac{z+3}{4}$ meets the plane $2x+4y-z+1=0$.
Ans. $(\frac{10}{3}, -\frac{3}{2}, \frac{5}{3})$.

Ex. 2. Find the points in which the line $\dfrac{x+1}{-1}=\dfrac{y-12}{5}=\dfrac{z-7}{2}$ cuts the surface $11x^2-5y^2+z^2=0$.
Ans. $(1, 2, 3), (2, -3, 1)$.

Ex. 3. If the axes are rectangular, find the distance from the point $(3, 4, 5)$ to the point where the line $\dfrac{x-3}{1}=\dfrac{y-4}{2}=\dfrac{z-5}{2}$ meets the plane $x+y+z=2$.
Ans. -6.

Ex. 4. Find the distance of the point $(1, -2, 3)$ from the plane $x-y+z=5$ measured parallel to the line $\dfrac{x}{2}=\dfrac{y}{3}=\dfrac{z}{-6}$, (rectangular axes).
Ans. 1.

Ex. 5. Shew that if the axes are rectangular, the equations to the perpendicular from the point (α, β, γ) to the plane $ax+by+cz+d=0$ are $\dfrac{x-\alpha}{a}=\dfrac{y-\beta}{b}=\dfrac{z-\gamma}{c}$, and deduce the perpendicular distance of the point (α, β, γ) from the plane.

Ex. 6. If the axes are rectangular, the equations to the line through (α, β, γ) at right angles to the lines

$$\frac{x}{l_1}=\frac{y}{m_1}=\frac{z}{n_1}, \quad \frac{x}{l_2}=\frac{y}{m_2}=\frac{z}{n_2}$$

are

$$\frac{x-\alpha}{m_1n_2-m_2n_1}=\frac{y-\beta}{n_1l_2-n_2l_1}=\frac{z-\gamma}{l_1m_2-l_2m_1}.$$

Ex. 7. If the axes are rectangular, shew that the equations to the planes through the lines which bisect the angles between

$$x/l_1=y/m_1=z/n_1 \quad \text{and} \quad x/l_2=y/m_2=z/n_2,$$

and at right angles to the plane containing them, are

$$(l_1 \pm l_2)x+(m_1 \pm m_2)y+(n_1 \pm n_2)z=0.$$

Ex. 8. A line through the origin makes angles α, β, γ with its projections on the coordinate planes, which are rectangular. The distances of any point (x, y, z) from the line and its projections are d, a, b, c. Prove that

$$d^2=(a^2-x^2)\cos^2\alpha+(b^2-y^2)\cos^2\beta+(c^2-z^2)\cos^2\gamma.$$

41. Line through two points. If P, Q are (x_1, y_1, z_1), (x_2, y_2, z_2), the direction-ratios of PQ are proportional to x_2-x_1, y_2-y_1, z_2-z_1, and therefore the equations to PQ are

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}.$$

By § 8, the coordinates of a variable point of the line in terms of one parameter, λ, are

$$x=\frac{\lambda x_2+x_1}{\lambda+1}, \quad y=\frac{\lambda y_2+y_1}{\lambda+1}, \quad z=\frac{\lambda z_2+z_1}{\lambda+1}.$$

Ex. 1. Find the point where the line joining (2, 1, 3), (4, −2, 5) cuts the plane $2x+y-z=3$. *Ans.* (0, 4, 1).

Ex. 2. Prove that the line joining the points (4, −5, −2), (−1, 5, 3) meets the surface $2x^2+3y^2-4z^2=1$ in coincident points.

42. Direction-ratios from equations. The planes through the origin parallel to

$$ax+by+cz+d=0, \quad a'x+b'y+c'z+d'=0$$

are given by

$$ax+by+cz=0, \quad a'x+b'y+c'z=0.$$

Hence the equations

$$ax+by+cz=0=a'x+b'y+c'z$$

together represent the straight line through the origin parallel to the line given by

$$ax+by+cz+d=0=a'x+b'y+c'z+d'.$$

They may be written

$$\frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{z}{ab'-a'b},$$

and therefore the direction-ratios of the two lines are proportional to $bc'-b'c$, $ca'-c'a$, $ab'-a'b$. Again the second line meets the plane $z=0$ in the point

$$\left(\frac{bd'-b'd}{ab'-a'b}, \frac{da'-d'a}{ab'-a'b}, 0\right);$$

therefore the equations to the second line in the symmetrical form are

$$\frac{x - \dfrac{bd' - b'd}{ab' - a'b}}{bc' - b'c} = \frac{y - \dfrac{da' - d'a}{ab' - a'b}}{ca' - c'a} = \frac{z}{ab' - a'b}.$$

Ex. 1. The equations to a line through (a, b, c) parallel to the plane **XOY** are

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{0}, \quad\ldots\ldots(1)$$

since the direction-ratios are l, m, 0. Again the line lies in the plane $z=c$, and therefore its equations can be written

$$m(x-a) = l(y-b),\ z=c, \quad\ldots\ldots(2)$$

and (1) is to be considered the symmetrical form of (2).

Ex. 2. Find the equations to the line joining $(2, 4, 3)$, $(-3, 5, 3)$. The equations are $\dfrac{x-2}{-5} = \dfrac{y-4}{1} = \dfrac{z-3}{0}$. Therefore the line is parallel to the plane **XOY**, as is evident, since the z-coordinates of two points on it are equal to 3. The equations can also be written

$$x + 5y = 22,\ z = 3.$$

Ex. 3. The equations to the straight line through (a, b, c) parallel to **OZ** are $\dfrac{x-a}{0} = \dfrac{y-b}{0} = \dfrac{z-c}{1}$ or $x=a$, $y=b$.

Ex. 4. Prove that the equations to the line of intersection of the planes $4x + 4y - 5z = 12$, $8x + 12y - 13z = 32$ can be written

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z}{4}.$$

Ex. 5. Shew that the line $2x + 2y - z - 6 = 0 = 2x + 3y - z - 8$ is parallel to the plane $y=0$, and find the coordinates of the point where it meets the plane $x=0$. *Ans.* $(0, 2, -2)$.

Ex. 6. Prove that the lines

$$2x + 3y - 4z = 0 = 3x - 4y + z, \quad 5x - y - 3z + 12 = 0 = x - 7y + 5z - 6$$

are parallel.

Ex. 7. Find the angle between the lines

$$x - 2y + z = 0 = x + y - z, \quad x + 2y + z = 0 = 8x + 12y + 5z,$$

(rectangular axes). *Ans.* $\cos^{-1} 8/\sqrt{406}$.

Ex. 8. Find the equations to the line through the point $(1, 2, 3)$ parallel to the line $x - y + 2z = 5$, $3x + y + z = 6$.

Ans. $\dfrac{x-1}{-3} = \dfrac{y-2}{5} = \dfrac{z-3}{4}$.

43. Constants in the equations to a line. The equations

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}$$

may be written

$$\left.\begin{aligned} x&=\frac{l}{m}y+a-\frac{lb}{m} \\ y&=\frac{m}{n}z+b-\frac{mc}{n} \end{aligned}\right\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

which are of the form

$$\left.\begin{aligned} x&=\mathsf{A}y+\mathsf{B} \\ y&=\mathsf{C}z+\mathsf{D} \end{aligned}\right\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

and therefore the general equations to a straight line contain four arbitrary constants. The equations (1) represent the planes passing through the line and parallel to OZ and OX respectively, and by a choice of such planes to define any given line its equations can be put in the form (2), which is the form with the smallest possible number of arbitrary constants.

Ex. 1. Prove that the symmetrical form of the equations to the line given by $x=ay+b$, $z=cy+d$ is $\dfrac{x-b}{a}=\dfrac{y}{1}=\dfrac{z-d}{c}$.

Ex. 2. Prove that the lines

$$x=ay+b, \quad z=cy+d, \quad x=a'y+b', \quad z=c'y+d',$$

are perpendicular if $aa'+cc'+1=0$.

Ex. 3. Find a, b, c, d, so that the line $x=ay+b$, $z=cy+d$ may pass through the points $(3, 2, -4)$, $(5, 4, -6)$, and hence shew that the given points and $(9, 8, -10)$ are collinear.

Ans. $a=1$, $b=1$, $c=-1$, $d=-2$.

Ex. 4. Prove that the line $x=pz+q$, $y=rz+s$, intersects the conic $z=0$, $ax^2+by^2=1$, if $aq^2+bs^2=1$.

Hence shew that the coordinates of any point on a line which intersects the conic and passes through the point (α, β, γ) satisfy the equation $a(\gamma x-\alpha z)^2+b(\gamma y-\beta z)^2=(z-\gamma)^2$.

Ex. 5. Prove that a line which passes through the point (α, β, γ) and intersects the parabola $y=0$, $z^2=4ax$, lies on the surface

$$(\beta z-\gamma y)^2=4a(\beta-y)(\beta x-\alpha y).$$

Ex. 6. Find the equations to the planes through the lines

(i) $\dfrac{x-2}{2}=\dfrac{y-3}{4}=\dfrac{z-4}{5}$, (ii) $2x+3y-5z-4=0=3x-4y+5z-6$,

parallel to the coordinate axes.

Ans. (i) $5y-4z+1=0$, $2z-5x+2=0$, $2x-y-1=0$;

(ii) $17y-25z=0$, $5z-17x+34=0$, $5x-y-10=0$.

***Ex. 7.** If the axes are oblique the distance of the point (x', y', z') from the plane $ax+by+cz+d=0$ is given by

$$\frac{\pm(ax'+by'+cz'+d)(1-\cos^2\lambda-\cos^2\mu-\cos^2\nu+2\cos\lambda\cos\mu\cos\nu)^{\frac{1}{2}}}{\{\Sigma a^2\sin^2\lambda-2\Sigma bc(\cos\lambda-\cos\mu\cos\nu)\}^{\frac{1}{2}}}.$$

***Ex. 8.** The distance of (x', y', z') from the line $x/a=y/b=z/c$ is given by

$$d^2=\frac{\Sigma(bz-cy)^2\sin^2\lambda+2\Sigma(cx-az)(ay-bx)(\cos\mu\cos\nu-\cos\lambda)}{a^2+b^2+c^2+2bc\cos\lambda+2ca\cos\mu+2ab\cos\nu}.$$

***Ex. 9.** Prove that the direction-cosines of the normal to the plane **OXY** are 0, 0, $\dfrac{\Delta^{\frac{1}{2}}}{\sin\nu}$,

where $$\Delta\equiv 1-\cos^2\lambda-\cos^2\mu-\cos^2\nu+2\cos\lambda\cos\mu\cos\nu.$$

If the angles that **OX, OY, OZ** make with the planes **YOZ, ZOX, XOY** are α, β, γ, prove that

$$\frac{\sin\alpha}{\operatorname{cosec}\lambda}=\frac{\sin\beta}{\operatorname{cosec}\mu}=\frac{\sin\gamma}{\operatorname{cosec}\nu}=\Delta^{\frac{1}{2}}.$$

If the angles between the planes **ZOX, XOY**, etc., are **A, B, C**, prove that

(i) $\cos\lambda-\cos\mu\cos\nu=\sin\mu\sin\nu\cos \mathrm{A}$,

(ii) $\dfrac{\sin \mathrm{A}}{\sin\lambda}=\dfrac{\sin \mathrm{B}}{\sin\mu}=\dfrac{\sin \mathrm{C}}{\sin\nu}$.

44. The plane and the straight line. Let the equations $ax+by+cz+d=0$, $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$ represent a given plane and straight line. Their point of intersection is

$$(\alpha+lr,\ \beta+mr,\ \gamma+nr),$$

where r is given by

$$r(al+bm+cn)+a\alpha+b\beta+c\gamma+d=0.$$

But r is proportional to the distance of the point from (α, β, γ). Therefore the line is parallel to the plane if

$$al+bm+cn=0 \text{ and } a\alpha+b\beta+c\gamma+d\neq 0.$$

If the axes are rectangular, the direction-cosines of the normal to the plane and of the line are proportional to a, b, c; l, m, n; and therefore if the line is normal to the plane,

$$\frac{l}{a}=\frac{m}{b}=\frac{n}{c}.$$

Cor. The conditions that the line should lie in the plane are

$$al+bm+cn=0$$

and

$$a\alpha+b\beta+c\gamma+d=0.$$

Ex. 1. Prove that the line $\frac{x-3}{2}=\frac{y-4}{3}=\frac{z-5}{4}$ is parallel to the plane $4x+4y-5z=0$.

Ex. 2. Prove that the planes $2x-3y-7z=0$, $3x-14y-13z=0$, $8x-31y-33z=0$ pass through one line.

Ex. 3. Find the equation to the plane through $(2, -3, 1)$ normal to the line joining $(3, 4, -1)$, $(2, -1, 5)$, (axes rectangular).

Ans. $x+5y-6z+19=0$.

Ex. 4. Find the equation to the plane through the points $(2, -1, 0)$, $(3, -4, 5)$ parallel to the line $2x=3y=4z$.

Ans. $29x-27y-22z=85$.

Ex. 5. Prove that the join of $(2, 3, 4)$, $(3, 4, 5)$ is normal to the plane through $(-2, -3, 6)$, $(4, 0, -3)$, $(0, -1, 2)$, the axes being rectangular.

Ex. 6. Find the distance of the point $(-1, -5, -10)$ from the point of intersection of the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ and the plane $x-y+z=5$, (rectangular axes). *Ans.* 13.

Ex. 7. Find the equations to the planes through the point $(-1, 0, 1)$ and the lines

$$4x-3y+1=0=y-4z+13\,; \quad 2x-y-2=0=z-5,$$

and shew that the equations to the line through the given point which intersects the two given lines can be written

$$x=y-1=z-2.$$

Ex. 8. Find the equation to the plane through the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

parallel to the line $$\frac{x}{l'}=\frac{y}{m'}=\frac{z}{n'}.$$

Ans. $\Sigma(x-\alpha)(mn'-m'n)=0$.

Ex. 9. The plane $lx+my=0$ is rotated about its line of intersection with the plane $z=0$ through an angle α. Prove that the equation to the plane in its new position is

$$lx+my\pm z\sqrt{l^2+m^2}\tan\alpha=0.$$

Ex. 10. Find the equations to the line through (f, g, h) which is parallel to the plane $lx+my+nz=0$ and intersects the line

$$ax+by+cz+d=0, \quad a'x+b'y+c'z+d'=0.$$

Ans. $$\begin{cases} l(x-f)+m(y-g)+n(z-h)=0, \\ \dfrac{ax+by+cz+d}{af+bg+ch+d}=\dfrac{a'x+b'y+c'z+d'}{a'f+b'g+c'h+d'}. \end{cases}$$

Ex. 11. The axes being rectangular, find the equations to the perpendicular from the origin to the line:

$$x+2y+3z+4=0,\quad 2x+3y+4z+5=0.$$

Find also the coordinates of the foot of the perpendicular.

(The perpendicular is the line of intersection of the plane through the origin and the line and the plane through the origin perpendicular to the line.)

Ans. $\dfrac{x}{2}=\dfrac{y}{-1}=\dfrac{z}{-4}$; $\left(\dfrac{2}{3}, \dfrac{-1}{3}, \dfrac{-4}{3}\right)$.

Ex. 12. The equations to AB referred to rectangular axes are $\dfrac{x}{2}=\dfrac{y}{-3}=\dfrac{z}{6}$. Through a point P, (1, 2, 5) PN is drawn perpendicular to AB, and PQ is drawn parallel to the plane $3x+4y+5z=0$ to meet AB in Q. Find the equations to PN and PQ and the coordinates of N and Q.

Ans. $\dfrac{x-1}{-3}=\dfrac{y-2}{176}=\dfrac{z-5}{89}$; $\dfrac{x-1}{4}=\dfrac{y-2}{-13}=\dfrac{z-5}{8}$;

$\left(\dfrac{52}{49}, \dfrac{-78}{49}, \dfrac{156}{49}\right)$; $\left(3, \dfrac{-9}{2}, 9\right)$.

Ex. 13. Through a point P, (x', y', z') a plane is drawn at right angles to OP to meet the axes (rectangular) in A, B, C. Prove that the area of the triangle ABC is $\dfrac{r^5}{2x'y'z'}$, where r is the measure of OP.

Ex. 14. The axes are rectangular and the plane $x/a+y/b+z/c=1$ meets them in A, B, C. Prove that the equations to BC are $\dfrac{x}{0}=\dfrac{y}{b}=\dfrac{z-c}{-c}$; that the equation to the plane through OX at right angles to BC is $by=cz$; that the three planes through OX, OY, OZ, at right angles to BC, CA, AB respectively, pass through the line $ax=by=cz$; and that the coordinates of the orthocentre of the triangle ABC are:

$$\frac{a^{-1}}{a^{-2}+b^{-2}+c^{-2}},\quad \frac{b^{-1}}{a^{-2}+b^{-2}+c^{-2}},\quad \frac{c^{-1}}{a^{-2}+b^{-2}+c^{-2}}.$$

Ex. 15. If the axes are rectangular, the distance of the point (x_0, y_0, z_0) from the line

$$u\equiv ax+by+cz+d=0,\quad v\equiv a'x+b'y+c'z+d'=0$$

is given by $$\left\{\frac{(a'u_0-av_0)^2+(b'u_0-bv_0)^2+(c'u_0-cv_0)^2}{(bc'-b'c)^2+(ca'-c'a)^2+(ab'-a'b)^2}\right\}^{\frac{1}{2}},$$

where $u_0\equiv ax_0+by_0+cz_0+d$, and $v_0\equiv a'x_0+b'y_0+c'z_0+d'$.

Ex. 16. Find the equation to the plane through the line

$$u\equiv ax+by+cz+d=0,\quad v\equiv a'x+b'y+c'z+d'=0,$$

parallel to the line $x/l=y/m=z/n$.

Ans. $u(a'l+b'm+c'n)=v(al+bm+cn)$.

Ex. 17. Find the equation to the plane through the lines

$$ax+by+cz=0=a'x+b'y+c'z, \quad \alpha x+\beta y+\gamma z=0=\alpha' x+\beta' y+\gamma' z.$$

Ans. $$\begin{vmatrix} x, & y, & z \\ bc'-b'c, & ca'-c'a, & ab'-a'b \\ \beta\gamma'-\beta'\gamma, & \gamma\alpha'-\gamma'\alpha, & \alpha\beta'-\alpha'\beta \end{vmatrix}=0.$$

Ex. 18. Prove that the plane through the point (α, β, γ) and the line $x=py+q=rz+s$ is given by

$$\begin{vmatrix} x, & py+q, & rz+s \\ \alpha, & p\beta+q, & r\gamma+s \\ 1, & 1, & 1 \end{vmatrix}=0.$$

Ex. 19. The distance of the point (ξ, η, ζ) from the line $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$, measured parallel to the plane $ax+by+cz=0$, is given by

$$d^2=\frac{(a^2+b^2+c^2)\Sigma\{m(\gamma-\zeta)-n(\beta-\eta)\}^2-\{\Sigma(\alpha-\xi)(bn-cm)\}^2}{(al+bm+cn)^2}.$$

Deduce the perpendicular distance of the point from the line.

***Ex. 20.** If the axes are oblique, the line $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$ is normal to the plane $ax+by+cz+d=0$, if

$$\frac{\dfrac{\partial\phi}{\partial l}}{a}=\frac{\dfrac{\partial\phi}{\partial m}}{b}=\frac{\dfrac{\partial\phi}{\partial n}}{c}. \quad \text{(See § 31.)}$$

***Ex. 21.** Shew that the equation to the plane through **OZ** at right angles to the plane **XOY** is

$$x(\cos\mu\cos\nu-\cos\lambda)=y(\cos\nu\cos\lambda-\cos\mu).$$

***Ex. 22.** Shew that the planes through **OX, OY, OZ,** at right angles to the planes **YOZ, ZOX, XOY,** pass through the line

$$x(\cos\mu\cos\nu-\cos\lambda)=y(\cos\nu\cos\lambda-\cos\mu)=z(\cos\lambda\cos\mu-\cos\nu).$$

***Ex. 23.** The planes through **O** normal to **OX, OY, OZ** cut the planes **YOZ, ZOX, XOY** in lines which lie in the plane

$$\frac{x}{\cos\lambda}+\frac{y}{\cos\mu}+\frac{z}{\cos\nu}=0.$$

***Ex. 24.** Shew that the line in Ex. 22 is at right angles to the plane in Ex. 23.

***Ex. 25.** If **P** is the point (x', y', z') and the perpendiculars from **P** to the coordinate planes are p_1, p_2, p_3, prove that

$$\frac{p_1\sin\lambda}{x'}=\frac{p_2\sin\mu}{y'}=\frac{p_3\sin\nu}{z'}=\Delta^{\frac{1}{2}}.$$

Deduce that the planes bisecting the interior angles between the coordinate planes pass through the line

$$\frac{x}{\sin\lambda}=\frac{y}{\sin\mu}=\frac{z}{\sin\nu}.$$

*Ex. 26. Shew that the squares of the distances of P, (x', y', z') from the coordinate axes are

$$y'^2 \sin^2\nu + z'^2 \sin^2\mu + 2y'z'(\cos\lambda - \cos\mu\cos\nu), \text{ etc.}$$

*Ex. 27. Prove that the equation to the plane through O normal to

$$\frac{x}{\sin\lambda} = \frac{y}{\sin\mu} = \frac{z}{\sin\nu}$$

is $$x\cos\frac{-\lambda+\mu+\nu}{2} + y\cos\frac{\lambda-\mu+\nu}{2} + z\cos\frac{\lambda+\mu-\nu}{2} = 0.$$

45. The intersection of three planes. Before proceeding to the general discussion of the intersection of three given planes we will consider three typical numerical cases.

Solving the equations

$$3x - 4y + 5z = 10,$$
$$2x - y + z = 3,$$
$$x - 3y + 2z = 1,$$

we obtain $x = 1$, $y = 2$, $z = 3$, and hence the three planes represented by the given equations pass through the point (1, 2, 3).

Let us now attempt to solve the equations

$$\text{(i)}\quad 2x - 4y + 2z = 5,$$
$$\text{(ii)}\quad 5x - y - z = 8,$$
$$\text{(iii)}\quad x + y - z = 7.$$

Eliminate z from (ii) and (iii), then from (i) and (ii), and we get $$4x - 2y = 1,\quad 4x - 2y = 7.$$

Whence subtracting, $0 \,.\, x + 0 \,.\, y = 6$.

Similarly, eliminating y from (i) and (ii), then from (ii) and (iii), we get

$$6x - 2z = 9,\quad 6x - 2z = 15,$$

whence $$0 \,.\, x + 0 \,.\, z = 6.$$

There are, therefore, no finite values of x, y, z, which satisfy all the given equations. The equations $0 \,.\, x + 0 \,.\, y = 6$, $0 \,.\, x + 0 \,.\, z = 6$, are limiting forms of $\frac{x}{k} + \frac{y}{k} = 6$, $\frac{x}{k} + \frac{z}{k} = 6$, as k tends to infinity, and hence we may say that any point whose coordinates satisfy the three given equations is at an

infinite distance. We easily find that the lines of intersection of any two of the planes are parallel to the line

$$\frac{x}{1}=\frac{y}{2}=\frac{z}{3},$$

and it is evident that no two of the planes are parallel, so that the three planes form a triangular prism. Thus if we are given the three equations to the faces of a triangular prism, and we attempt to solve them, we obtain a paradoxical equation of the form $k=0$, where k is a number different from zero.

Consider, in the third place, the equations

$$\begin{aligned} &\text{(i)} & 12x-\ y+2z&=35,\\ &\text{(ii)} & 3x+\ y+\ z&=\ 7,\\ &\text{(iii)} & x+2y+\ z&=\ 0. \end{aligned}$$

Eliminating z between (i) and (ii), and then between (ii) and (iii), we obtain

$$6x-3y=21, \quad 2x-y=7.$$

Similarly, if we eliminate x in any way between the equations, we get $\quad 5y+2z+7=0.$

Thus all points whose coordinates satisfy the given equations lie upon both of the planes $2x-y=7$, $5y+2z+7=0$, or the common points of the three planes lie upon a straight line, that is, the three planes intersect in a straight line.

Ex. 1. Examine the nature of the intersection of the sets of planes :

(i) $2x-5y+\ z=3,\quad x+\ y+4z=5,\quad x+3y+6z=1$;
(ii) $3x+4y+6z=5,\quad 6x+5y+9z=10,\quad 3x+3y+5z=5$;
(iii) $x+\ y+\ z=6,\quad 2x+3y+4z=20,\quad x-\ y+\ z=2$;
(iv) $x+2y+3z=6,\quad 3x+4y+5z=2,\quad 5x+4y+3z+18=0$;
(v) $2x+3y+4z=6,\quad 3x+4y+5z=20,\quad x+2y+3z=2$;
(vi) $2x-\ y+\ z=4,\quad 5x+7y+2z=0,\quad 3x+4y-2z+3=0$;
(vii) $3x-\ y+\ z=5,\quad 2x+4y+\ z+10=0,\quad 6x-2y+2z+9=0.$

Ans. (i) Planes form prism ; (ii) planes pass through line

$$\frac{3x-5}{2}=\frac{y}{1}=\frac{z}{-1};$$

(iii) planes intersect at (1, 2, 3) ; (iv) planes pass through line

$$\frac{x+10}{1}=\frac{y-8}{-2}=\frac{z}{1};$$

(v) planes form prism ; (vi) planes intersect at $(1, -1, 1)$; (vii) two planes parallel, third intersects them.

Ex. 2. Prove that the three planes $2x+y+z=3$, $x-y+2z=4$, $x+z=2$, form a triangular prism, and find the area of a normal section of the prism. *Ans.* $\sqrt{3}/18$.

We shall now consider the general case.

Let the equations to the planes be

$$u_1 \equiv a_1x+b_1y+c_1z+d_1=0, \quad \dots\dots\dots\dots\dots(1)$$
$$u_2 \equiv a_2x+b_2y+c_2z+d_2=0, \quad \dots\dots\dots\dots\dots(2)$$
$$u_3 \equiv a_3x+b_3y+c_3z+d_3=0. \quad \dots\dots\dots\dots\dots(3)$$

Solving the equations (1), (2), (3), we obtain:

$$\frac{x}{\begin{vmatrix} b_1, & c_1, & d_1 \\ b_2, & c_2, & d_2 \\ b_3, & c_3, & d_3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1, & c_1, & d_1 \\ a_2, & c_2, & d_2 \\ a_3, & c_3, & d_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1, & b_1, & d_1 \\ a_2, & b_2, & d_2 \\ a_3, & b_3, & d_3 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}},$$

which, by contracting the denominators, we may write,

$$\frac{x}{|b_1, c_2, d_3|} = \frac{-y}{|a_1, c_2, d_3|} = \frac{z}{|a_1, b_2, d_3|} = \frac{-1}{|a_1, b_2, c_3|}. \quad \dots(4)$$

Let $\Delta \equiv |a_1, b_2, c_3|$, $\mathsf{A}_1 \equiv \dfrac{\partial\Delta}{\partial a_1} \equiv (b_2c_3-b_3c_2)$,

$$\mathsf{B}_1 \equiv \frac{\partial\Delta}{\partial b_1} \equiv (c_2a_3-c_3a_2), \quad \mathsf{C}_1 \equiv \frac{\partial\Delta}{\partial c_1} \equiv (a_2b_3-a_3b_2); \text{ etc.}$$

Then

$$\mathsf{B}_2\mathsf{C}_3-\mathsf{B}_3\mathsf{C}_2=a_1\Delta, \quad \mathsf{C}_2\mathsf{A}_3-\mathsf{C}_3\mathsf{A}_2=b_1\Delta, \quad \mathsf{A}_2\mathsf{B}_3-\mathsf{A}_3\mathsf{B}_2=c_1\Delta; \text{ etc.}$$

Therefore, if $\Delta=0$,

$$\frac{\mathsf{A}_2}{\mathsf{A}_3}=\frac{\mathsf{B}_2}{\mathsf{B}_3}=\frac{\mathsf{C}_2}{\mathsf{C}_3} \text{ and } \frac{\mathsf{A}_3}{\mathsf{A}_1}=\frac{\mathsf{B}_3}{\mathsf{B}_1}=\frac{\mathsf{C}_3}{\mathsf{C}_1}. \quad \dots\dots\dots\dots(5)$$

If $\Delta \neq 0$, the equations (4) give finite values of x, y, z, and therefore the three given planes have a point of intersection at a finite distance.

But if $\Delta=0$ and $|b_1, c_2, d_3| \neq 0$, the given equations are not satisfied by any finite value of x. Since

$$|b_1, c_2, d_3| \equiv d_1\mathsf{A}_1+d_2\mathsf{A}_2+d_3\mathsf{A}_3,$$

A_1, A_2, A_3 cannot all be zero, and therefore the three planes are not all parallel. Again, the lines of intersection of the planes are parallel to

$$\frac{x}{\mathsf{A}_1}=\frac{y}{\mathsf{B}_1}=\frac{z}{\mathsf{C}_1}, \quad \frac{x}{\mathsf{A}_2}=\frac{y}{\mathsf{B}_2}=\frac{z}{\mathsf{C}_2}, \quad \frac{x}{\mathsf{A}_3}=\frac{y}{\mathsf{B}_3}=\frac{z}{\mathsf{C}_3},$$

and hence, by (5), are parallel. If two or more of the quantities A_1, A_2, A_3 are different from zero, no two of the given planes are parallel, and the planes therefore form a triangular prism. If one only, A_1 say, of the three quantities is different from zero, the planes $u_1=0$, $u_2=0$ may be parallel, and if so, $u_3=0$ meets them in parallel lines. We have then a limiting case of a triangular prism when one of the edges is at an infinite distance. Thus, if $\Delta=0$ and $|b_1, c_2, d_3|\neq 0$, the three planes are parallel to one line.

It is to be noted that in this case

$$A_1u_1+A_2u_2+A_3u_3\equiv|b_1, c_2, d_3|\neq 0,$$

that is, when three planes are parallel to one line their equations can be combined so as to form a paradoxical equation $k=0$, where k is a quantity different from zero. Conversely, if three numbers l, m, n can be found so that

$$lu_1+mu_2+nu_3\equiv k,$$

where k is independent of x, y, z, and is not zero, then the three planes are parallel to one line, and if no two of them are parallel, form a triangular prism. For

$$a_1l+a_2m+a_3n=0, \quad b_1l+b_2m+b_3n=0,$$
$$c_1l+c_2m+c_3n=0, \quad d_1l+d_2m+d_3n\neq 0.$$

Therefore $|a_1, b_2, c_3|=0$ and $|b_1, c_2, d_3|\neq 0$.

Suppose now that $\Delta=0$, $|b_1, c_2, d_3|=0$ and $A_1\neq 0$, (A_1 is one of the common minors of Δ and $|b_1, c_2, d_3|$). As in the last case, the three planes are parallel to one line. But since $|b_1, c_2, d_3|=0$, the three lines in which the planes cut the plane YOZ, viz.,

$$x=0, \quad b_1y+c_1z+d_1=0;$$
$$x=0, \quad b_2y+c_2z+d_2=0;$$
$$x=0, \quad b_3y+c_3z+d_3=0$$

are concurrent. Their common point is given by $x=0$,

$$\frac{y}{c_2d_3-c_3d_2}=\frac{z}{d_2b_3-d_3b_2}=\frac{1}{A_1},$$

and since $A_1\neq 0$, it is at a finite distance. Hence, since the

three planes are parallel to one line and pass through a point in the plane YOZ, they pass through one line.

It follows now that $|a_1, c_2, d_3|$ and $|a_1, b_2, d_3|$, the remaining two determinants in (4), are zero. For since the planes pass through one line, their lines of intersection with the plane ZOX, viz.,

$$y=0,\ \ a_1x+c_1z+d_1=0;\quad y=0,\ \ a_2x+c_2z+d_2=0;$$
$$y=0,\ \ a_3x+c_3z+d_3=0$$

are concurrent. Therefore $|a_1, c_2, d_3|=0$, and similarly, $|a_1, b_2, d_3|=0$.

Again, if $|a_1, b_2, d_3|=0$, $|a_1, c_2, d_3|=0$ and $a_2d_3-a_3d_2$, (any one of the common minors), is not zero, the lines of intersection of the given planes with the planes ZOX and XOY are concurrent. The points of concurrence are given by

$$y=0,\ \ \frac{x}{c_2d_3-c_3d_2}=\frac{z}{d_2a_3-d_3a_2}=\frac{1}{a_2c_3-a_3c_2};$$
$$z=0,\ \ \frac{x}{b_2d_3-b_3d_2}=\frac{y}{d_2a_3-d_3a_2}=\frac{1}{a_2b_3-b_3a_2};$$

and since $d_2a_3-d_3a_2 \neq 0$, they are not coincident. The planes have therefore two common points and thus pass through one line. It follows then that $|a_1, b_2, c_3|$ and $|b_1, c_2, d_3|$ are both zero.

If, therefore, any two of the determinants

$$|b_1, c_2, d_3|,\quad |a_1, c_2, d_3|,\quad |a_1, b_2, d_3|,\quad |a_1, b_2, c_3|$$

are zero, and one of their common minors is not zero, the remaining two determinants are zero,* and the three planes have a line of intersection at a finite distance.

---

* This is easily proved algebraically. If $\Delta=0$, $|b_1, c_2, d_3|=0$, and $A_1 \neq 0$, then, since

$$\Delta \equiv a_1A_1+a_2A_2+a_3A_3=0,$$

and

$$|b_1, c_2, d_3| \equiv d_1A_1+d_2A_2+d_3A_3=0,$$

we have

$$\frac{A_1}{a_2d_3-a_3d_2}=\frac{A_2}{a_3d_1-a_1d_3}=\frac{A_3}{a_1d_2-a_2d_1}$$
$$=\frac{b_1A_1+b_2A_2+b_3A_3}{-|a_1, b_2, d_3|}=\frac{c_1A_1+c_2A_2+c_3A_3}{-|a_1, c_2, d_3|}.$$

Therefore, since $\Sigma b_1A_1 \equiv 0$, $\Sigma c_1A_1 \equiv 0$, and $A_1 \neq 0$,

$$|a_1, b_2, d_3|=0 \text{ and } |a_1, c_2, d_3|=0.$$

The conditions for a line of intersection are often written in the form,

$$\left\| \begin{matrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \end{matrix} \right\| = 0,$$

the notation signifying that any two of the four third-order determinants are zero. They may also be obtained as follows. Any plane through the line of intersection of $u_1=0$, $u_2=0$ is given by $\lambda_1 u_1+\lambda_2 u_2=0$. If the planes $u_1=0$, $u_2=0$, $u_3=0$ pass through one line,

$$\lambda_1 u_1+\lambda_2 u_2=0 \quad \text{and} \quad u_3=0$$

must, for some values of λ_1, λ_2, represent the same plane, and therefore

$$\lambda_1 u_1+\lambda_2 u_2 \equiv -\lambda_3 u_3,$$

or
$$\lambda_1 u_1+\lambda_2 u_2+\lambda_3 u_3 \equiv 0.$$

Conversely, if λ_1, λ_2, λ_3 can be found so that

$$\lambda_1 u_1+\lambda_2 u_2+\lambda_3 u_3 \equiv 0,$$

then
$$\lambda_1 u_1+\lambda_2 u_2 \equiv -\lambda_3 u_3,$$

and therefore the plane $u_3=0$ passes through the line of intersection of $u_1=0$ and $u_2=0$. Considering the coefficients in $\lambda_1 u_1+\lambda_2 u_2+\lambda_3 u_3 \equiv 0$, we have

$$a_1\lambda_1+a_2\lambda_2+a_3\lambda_3=0, \qquad b_1\lambda_1+b_2\lambda_2+b_3\lambda_3=0,$$
$$c_1\lambda_1+c_2\lambda_2+c_3\lambda_3=0, \quad \text{and} \quad d_1\lambda_1+d_2\lambda_2+d_3\lambda_3=0.$$

Therefore, eliminating λ_1, λ_2, λ_3, we obtain

$$\left\| \begin{matrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \end{matrix} \right\| = 0.$$

Ex. 3. Prove that the planes

$$x+ay+(b+c)z+d=0,$$
$$x+by+(c+a)z+d=0,$$
$$x+cy+(a+b)z+d=0$$

pass through one line.

Ex. 4. Prove that the planes $x=cy+bz$, $y=az+cx$, $z=bx+ay$ pass through one line if $a^2+b^2+c^2+2abc=1$.

Ex. 5. The planes $ax+hy+gz=0$, $hx+by+fz=0$, $gx+fy+cz=0$ pass through one line if $\Delta \equiv \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = 0$, and the direction-ratios of the line satisfy the equations

$$\frac{l^2}{\frac{\partial\Delta}{\partial a}} = \frac{m^2}{\frac{\partial\Delta}{\partial b}} = \frac{n^2}{\frac{\partial\Delta}{\partial c}}.$$

Ex. 6. If the axes are rectangular, the equations to the planes through the line of intersection of two of the given planes

$$a_r x + b_r y + c_r z + d_r = 0, \quad r = 1, 2, 3,$$

perpendicular to the third, are

$$(a_1 x + b_1 y + c_1 z + d_1)(a_2 a_3 + b_2 b_3 + c_2 c_3) - (a_2 x + b_2 y + c_2 z + d_2) \times (a_3 a_1 + b_3 b_1 + c_3 c_1) = 0, \text{ etc.}$$

Shew that the three planes pass through one line.

Ex. 7. The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the axes OX, OY, OZ, which are rectangular, in A, B, C. Prove that the planes through the axes and the internal bisectors of the angles of the triangle ABC pass through the line

$$\frac{x}{a\sqrt{b^2+c^2}} = \frac{y}{b\sqrt{c^2+a^2}} = \frac{z}{c\sqrt{a^2+b^2}}.$$

46. Line intersecting two given lines. The equations to any line intersecting two given lines, $u_1 = 0 = v_1$; $u_2 = 0 = v_2$, are

$$u_1 + \lambda_1 v_1 = 0, \quad u_2 + \lambda_2 v_2 = 0.$$

For the third line lies in the plane $u_1 + \lambda_1 v_1 = 0$, and therefore it is coplanar with $u_1 = 0 = v_1$, and similarly it is coplanar with $u_2 = 0 = v_2$.

Ex. 1. Find the equations to the straight line drawn from the origin to intersect the lines

$$3x+2y+4z-5=0=2x-3y+4z+1,$$
$$2x-4y+z+6=0=3x-4y+z-3.$$

Ans. $\frac{x}{249}=\frac{y}{153}=\frac{z}{-52}$.

Ex. 2. Find the equations to the line that intersects the lines $x+y+z=1$, $2x-y-z=2$; $x-y-z=3$, $2x+4y-z=4$, and passes through the point (1, 1, 1). *Ans.* $\frac{x-1}{0}=\frac{y-1}{1}=\frac{z-1}{3}$.

Ex. 3. Find the equations to the line drawn parallel to $\frac{x}{4}=\frac{y}{1}=\frac{z}{1}$ so as to meet the lines $z=5x-6=4y+3$, $z=2x-4=3y+5$.
Ans. $44z=11x+1693$, $11z=11y+345$.

Ex. 4. Find the surface generated by a line which intersects the lines $y=z=a$; $x+3z=a$, $y+z=a$, and is parallel to the plane $x+y=0$.
Ans. $(x+y)(y+z)=2a(z+x)$.

Ex. 5. Find the surface generated by a straight line which intersects the lines $x+y=z=0$; $x-y=z$, $x+y=2a$, and the parabola $y=0$, $x^2=2az$. *Ans.* $x^2-y^2=2az$.

Ex. 6. A variable line intersects OX, and the curve $x=y$, $y^2=cz$, and is parallel to the plane YOZ. Prove that it generates the paraboloid $xy=cz$.

Ex. 7. Prove that the locus of a variable line which intersects the three given lines $y=mx$, $z=c$; $y=-mx$, $z=-c$; $y=z$, $mx=-c$; is the surface $y^2-m^2x^2=z^2-c^2$.

47. Lines intersecting three given lines. If the equations to three given lines are $u_1=0=v_1$, $u_2=0=v_2$, $u_3=0=v_3$, and the three planes

$$(1)\ u_1-\lambda_1 v_1=0,\quad (2)\ u_2-\lambda_2 v_2=0,\quad (3)\ u_3-\lambda_3 v_3=0$$

have a line of intersection, that line is coplanar with each of the three given lines, and therefore intersects all three. There are two independent conditions for a line of intersection, (§ 45), which may be written,

$$f_1(\lambda_1,\ \lambda_2,\ \lambda_3)=0, \quad\ldots\ldots\ldots(4)\qquad f_2(\lambda_1,\ \lambda_2,\ \lambda_3)=0. \quad\ldots\ldots\ldots(5)$$

If λ_1, λ_2, λ_3 be chosen to satisfy (4) and (5), any two of the equations (1), (2), (3) represent a line which intersects the three given lines. Suppose that (1) and (2) are taken, then eliminating λ_3 between (4) and (5), we obtain

$$\phi(\lambda_1,\ \lambda_2)=0. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

An infinite number of values of λ_1, λ_2 can be found to satisfy (6), and therefore an infinite number of lines can be found to intersect three given lines. If we eliminate λ_1, λ_2 between (1), (2), (6) we obtain

$$\phi\left(\frac{u_1}{v_1},\ \frac{u_2}{v_2}\right)=0. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

This equation is satisfied by the coordinates of any point on any line which intersects the three given lines, and therefore represents a surface generated by such lines. Hence the lines which intersect three given lines lie on a surface.

It is to be noted that if λ_1, λ_2, λ_3 satisfy (4) and (5), (3) is of the form $u_1-\lambda_1 v_1+k(u_2-\lambda_2 v_2)=0$, and therefore that (1), (2), (3), (4), (5) are really equivalent to four independent equations. The equation to the surface is obtained by eliminating λ_1, λ_2, λ_3 between these four equations, and this can be done in only *one* way. Hence the surface is also given by $f_1\left(\frac{u_1}{v_1}, \frac{u_2}{v_2}, \frac{u_3}{v_3}\right)=0$, or by $f_2\left(\frac{u_1}{v_1}, \frac{u_2}{v_2}, \frac{u_3}{v_3}\right)=0$.

Ex. 1. Find the locus of lines which intersect the three lines $y=b$, $z=-c$; $z=c$; $x=-a$; $x=a$, $y=-b$.

If the three planes

$$y-b+\lambda_1(z+c)=0, \quad z-c+\lambda_2(x+a)=0, \quad x-a+\lambda_3(y+b)=0$$

have a line of intersection, it meets the three given lines. That is so if

$$\left\|\begin{matrix} 0, & 1, & \lambda_1, & -b+\lambda_1 c \\ \lambda_2, & 0, & 1, & -c+\lambda_2 a \\ 1, & \lambda_3, & 0, & -a+\lambda_3 b \end{matrix}\right\|=0,$$

i.e. if (1) $\lambda_1\lambda_2\lambda_3+1=0$ and (2) $\lambda_1\lambda_2\lambda_3 a-2c\lambda_1\lambda_3+2b\lambda_3-a=0$.

Therefore the coordinates of any point on a line which meets the three given lines satisfy

$$y-b+\lambda_1(z+c)=0, \quad z-c+\lambda_2(x+a)=0, \quad x-a+\lambda_3(y+b)=0,$$

where $\lambda_1\lambda_2\lambda_3+1=0$. Therefore eliminating λ_1, λ_2, λ_3, we obtain the locus of the lines, viz.:

$$\frac{y-b}{z+c}\cdot\frac{z-c}{x+a}\cdot\frac{x-a}{y+b}=1,$$

$$\text{or} \quad ayz+bzx+cxy+abc=0.$$

(Shew that the same result is obtained from (2).)

Ex. 2. If the planes through a point P and the three given lines $y=1$, $z=-1$; $z=1$, $x=-1$; $x=1$, $y=-1$ pass through one line, P lies on the surface $yz+zx+xy+1=0$.

Ex. 3. Prove that all lines which intersect the lines $y=mx$, $z=c$; $y=-mx$, $z=-c$; and the x-axis, lie on the surface $mxz=cy$.

Ex. 4. Prove that the locus of lines which intersect the three lines $y-z=1$, $x=0$; $z-x=1$, $y=0$; $x-y=1$, $z=0$ is

$$x^2+y^2+z^2-2yz-2zx-2xy=1.$$

Ex. 5. Find the locus of the straight lines which meet the lines

$$x=2,\ 4y=3z; \quad x+2=0,\ 4y+3z=0; \quad y=3,\ 2x+z=0.$$

Ans. $36x^2+16y^2-9z^2=144$.

Ex. 6. Shew that the equations to any line which intersects the three given lines $y=b$, $z=-c$, $z=c$, $x=-a$; $x=a$, $y=-b$ may be written $y-b+\lambda(z+c)=0$, $(x-a)+\mu(y+b)=0$, where λ and μ are connected by the equation $\lambda\mu c-\mu b+a=0$. Hence shew that the two lines which intersect the three given lines and also $\frac{x}{c}=\frac{y+c}{c}=\frac{z-b}{-(a+b)}$ are

$$\frac{x}{a}=\frac{y+c}{c-b}=\frac{z-b}{b-c}, \quad \frac{x-c}{c-a}=\frac{y}{b}=\frac{z-a}{a-c}.$$

Ex. 7. Shew that the two lines that can be drawn to intersect the four given lines

$$y=1,\ z=-1\ ;\quad z=1,\ x=-1\ ;\quad x=1,\ y=-1\ ;\quad x=0,\ y+z=0$$

are given by $z=1,\ y+1=0$; $z+2x+1=0,\ y-z-2=0$.

48. Coplanar lines. *To find the condition that two given lines should be coplanar.*

Let their equations be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}, \quad \text{.....................(1)}$$

$$\frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}. \quad \text{.....................(2)}$$

The equation to a plane through the first line is

$$a(x-\alpha)+b(y-\beta)+c(z-\gamma)=0, \quad \text{..............(3)}$$

where $$al+bm+cn=0. \quad \text{.........................(4)}$$

If it contains the line (2),

$$a(\alpha-\alpha')+b(\beta-\beta')+c(\gamma-\gamma')=0, \quad \text{............(5)}$$

and $$al'+bm'+cn'=0. \quad \text{.........................(6)}$$

Therefore eliminating a, b, c between (4), (5), (6), we obtain the required condition,

$$\begin{vmatrix} \alpha-\alpha', & \beta-\beta', & \gamma-\gamma' \\ l, & m, & n \\ l', & m', & n' \end{vmatrix}=0. \quad \text{..............(7)}$$

The elimination of a, b, c between (3), (4), (6) gives the equation to the plane containing the lines, viz.,

$$\begin{vmatrix} x-\alpha, & y-\beta, & z-\gamma \\ l, & m, & n \\ l', & m', & n' \end{vmatrix}=0. \quad \text{...............(8)}$$

Generally, the equation (8) represents the plane through the line (1) parallel to the line (2), and (7) is the condition that this plane should contain the point $(\alpha', \beta', \gamma')$ on (2).

Ex. 1. Deduce the result (7) by equating the coordinates $\alpha+lr$, etc., $\alpha'+l'r'$, etc., of variable points on the given lines.

Ex. 2. Prove that the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$; $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ are coplanar.

Ex. 3. Prove that the lines

$$\frac{x-a+d}{\alpha-\delta}=\frac{y-a}{\alpha}=\frac{z-a-d}{\alpha+\delta};$$

$$\frac{x-b+c}{\beta-\gamma}=\frac{y-b}{\beta}=\frac{z-b-c}{\beta+\gamma}$$

are coplanar, and find the equation to the plane in which they lie.
Ans. $2y=x+z$.

Ex. 4. Prove that the lines $x=ay+b=cz+d$, $x=\alpha y+\beta=\gamma z+\delta$ are coplanar if $(\gamma-c)(a\beta-b\alpha)-(\alpha-a)(c\delta-d\gamma)=0$.

Ex. 5. Prove that the lines

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}, \quad ax+by+cz+d=0=a'x+b'y+c'z+d'$$

are coplanar if $\dfrac{a\alpha+b\beta+c\gamma+d}{al+bm+cn}=\dfrac{a'\alpha+b'\beta+c'\gamma+d'}{a'l+b'm+c'n}$.

Ex. 6. Prove that the lines $ax+by+cz+d=0=a'x+b'y+c'z+d'$; $\alpha x+\beta y+\gamma z+\delta=0=\alpha' x+\beta' y+\gamma' z+\delta'$ are coplanar if

$$\begin{vmatrix} a, & a', & \alpha, & \alpha' \\ b, & b', & \beta, & \beta' \\ c, & c', & \gamma, & \gamma' \\ d, & d', & \delta, & \delta' \end{vmatrix}=0.$$

Ex. 7. A, A′; B, B′; C, C′ are points on the axes; shew that the lines of intersection of the planes A′BC, AB′C′; B′CA, BC′A′; C′AB, CA′B′ are coplanar.

49. The shortest distance between two lines. *The axes being rectangular, to find the shortest distance between the lines* $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$; $\dfrac{x-\alpha'}{l'}=\dfrac{y-\beta'}{m'}=\dfrac{z-\gamma'}{n'}$.

Let the points A, A′, (fig. 24), be (α, β, γ), $(\alpha', \beta', \gamma')$. The shortest distance between the given lines is at right angles to both, and it is therefore equal to the projection of AA′ on a line at right angles to both of the given lines.

Suppose that λ, μ, ν are the direction-cosines of such a line, then $l\lambda+m\mu+n\nu=0$ and $l'\lambda+m'\mu+n'\nu=0$;

$$\therefore \quad \frac{\lambda}{mn'-m'n}=\frac{\mu}{nl'-n'l}=\frac{\nu}{lm'-l'm}.$$

Therefore the projection

$$=\lambda(\alpha-\alpha')+\mu(\beta-\beta')+\nu(\gamma-\gamma'), \quad (\S\,21, \text{ Ex. } 3),$$

$$=\frac{(\alpha-\alpha')(mn'-m'n)+(\beta-\beta')(nl'-n'l)+(\gamma-\gamma')(lm'-l'm)}{\sqrt{\Sigma(mn'-m'n)^2}},$$

$$=\begin{vmatrix} \alpha-\alpha', & \beta-\beta', & \gamma-\gamma' \\ l, & m, & n \\ l', & m', & n' \end{vmatrix} \div \sqrt{\Sigma(mn'-m'n)^2}.$$

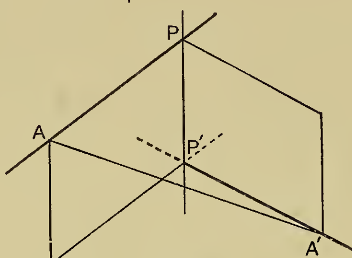

FIG. 24.

Equation (7), § 48, may now be interpreted as the condition that the shortest distance between the two given lines should vanish. Again, if PP′ is the shortest distance, the equations to the planes APP′, A′PP′ are

$$\begin{vmatrix} x-\alpha, & y-\beta, & z-\gamma \\ l, & m, & n \\ \lambda, & \mu, & \nu \end{vmatrix}=0, \quad \begin{vmatrix} x-\alpha', & y-\beta', & z-\gamma' \\ l', & m', & n' \\ \lambda, & \mu, & \nu \end{vmatrix}=0,$$

and these represent the line PP′.

Ex. 1. Find the shortest distance using the theorems that the shortest distance is equal to (i), the perpendicular from any point $(\alpha+lr,\ \beta+mr,\ \gamma+nr)$, on the first line to the plane drawn through the second parallel to the first; and (ii), the distance between two planes, each passing through one line and parallel to the other.

Ex. 2. If P, $(\alpha+lr,\ \beta+mr,\ \gamma+nr)$ and

P′, $(\alpha'+l'r',\ \beta'+m'r',\ \gamma'+n'r')$

are points on the given lines, and $\mathrm{PP}'=\delta$, prove that $\dfrac{\partial\delta^2}{\partial r}=0$, $\dfrac{\partial\delta^2}{\partial r'}=0$, necessary conditions for a minimum of PP'^2, are verified when PP′ is perpendicular to each of the lines.

Ex. 3. Shew that the shortest distance between the lines

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}; \quad \frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$$

is $\frac{1}{\sqrt{6}}$, and that its equations are

$$11x+2y-7z+6=0, \quad 7x+y-5z+7=0.$$

Ex. 4. Find the shortest distance between the lines

$$\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}; \quad \frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}.$$

The following method of solution may be adopted: Let the S.D. meet the lines in P and P′ respectively. Then the coordinates of P and P′ may be written $(3+3r, 8-r, 3+r)$, $(-3-3r', -7+2r', 6+4r')$, where r is proportional to the distance of P from the point (3, 8, 3) and r' to the distance of P′ from (−3, −7, 6). Whence the direction-cosines of PP′ are proportional to $6+3r+3r'$, $15-r-2r'$, $-3+r-4r'$.

Since PP′ is at right angles to both lines, we have

$$3(6+3r+3r')-(15-r-2r')+(-3+r-4r')=0,$$

$$-3(6+3r+3r')+2(15-r-2r')+4(-3+r-4r')=0.$$

Whence, solving for r and r', we get $r=r'=0$.

Therefore P and P′ are the points (3, 8, 3), (−3, −7, 6), $PP'=3\sqrt{30}$, and the equations to PP′ are

$$\frac{x-3}{2}=\frac{y-8}{5}=\frac{z-3}{-1}.$$

Ex. 5. Find the same results for the lines

$$\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}; \quad \frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}.$$

Ans. $2\sqrt{29}$, $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$, (3, 5, 7), (−1, −1, −1).

Ex. 6. Find the length and equations of the S.D. between

$$3x-9y+5z=0=x+y-z,$$

$$6x+8y+3z-13=0=x+2y+z-3.$$

Ans. $\frac{11}{\sqrt{342}}$, $10x-29y+16z=0=13x+82y+55z-109$.

Ex. 7. A line with direction-cosines proportional to 2, 7, −5 is drawn to intersect the lines

$$\frac{x-5}{3}=\frac{y-7}{-1}=\frac{z+2}{1}; \quad \frac{x+3}{-3}=\frac{y-3}{2}=\frac{z-6}{4}.$$

Find the coordinates of the points of intersection and the length intercepted on it. *Ans.* (2, 8, −3), (0, 1, 2), $\sqrt{78}$.

Ex. 8. Find the S.D. between the axis of z and the line

$$ax+by+cz+d=0, \quad a'x+b'y+c'z+d'=0.$$

(The plane passing through the line and parallel to **OZ** is

$$c'(ax+by+cz+d)=c(a'x+b'y+c'z+d'),$$

and the perpendicular from the origin to this plane is equal to the S.D.)

Ans. $\dfrac{cd'-c'd}{\sqrt{(ac'-a'c)^2+(bc'-b'c)^2}}.$

Ex. 9. If the axes are rectangular, the S.D. between the lines $y=az+b$, $z=\alpha x+\beta$; $y=a'z+b'$, $z=\alpha' x+\beta'$ is

$$\frac{(\alpha-\alpha')(b-b')+(\alpha'\beta-\alpha\beta')(a-a')}{\{\alpha^2\alpha'^2(a-a')^2+(\alpha-\alpha')^2+(a\alpha-a'\alpha')^2\}^{\frac{1}{2}}}.$$

Ex. 10. Prove that the S.D. between the lines

$$ax+by+cz+d=0=a'x+b'y+c'z+d',$$

$$\alpha x+\beta y+\gamma z+\delta=0=\alpha' x+\beta' y+\gamma' z+\delta'$$

is

$$\begin{vmatrix} a, & b, & c, & d \\ a', & b', & c', & d' \\ \alpha, & \beta, & \gamma, & \delta \\ \alpha', & \beta', & \gamma', & \delta' \end{vmatrix} \div \{\Sigma(\mathrm{BC}'-\mathrm{B}'\mathrm{C})^2\}^{\frac{1}{2}},$$

where $\mathrm{A}\equiv bc'-b'c$, etc., $\mathrm{A}'\equiv\beta\gamma'-\beta'\gamma$, etc.

Ex. 11. Shew that the S.D. between the lines

$$\frac{x-x_1}{\cos\alpha_1}=\frac{y-y_1}{\cos\beta_1}=\frac{z-z_1}{\cos\gamma_1}; \quad \frac{x-x_2}{\cos\alpha_2}=\frac{y-y_2}{\cos\beta_2}=\frac{z-z_2}{\cos\gamma_2}$$

meets the first line at a point whose distance from (x_1, y_1, z_1) is $\dfrac{\Sigma(x_1-x_2)(\cos\alpha_1-\cos\theta\cos\alpha_2)}{\sin^2\theta}$, where θ is the angle between the lines.

Ex. 12. Shew that the S.D. between any two opposite edges of the tetrahedron formed by the planes $y+z=0$, $z+x=0$, $x+y=0$, $x+y+z=a$ is $2a/\sqrt{6}$, and that the three lines of shortest distance intersect at the point $x=y=z=a$.

Ex. 13. Shew that the S.D. between the line

$$ax+by+cz+d=0=a'x+b'y+c'z+d'$$

and the z-axis meets the z-axis at a point whose distance from the origin is

$$\frac{(db'-d'b)(bc'-b'c)+(ca'-c'a)(ad'-a'd)}{\{(bc'-b'c)^2+(ca'-c'a)^2\}}.$$

Ex. 14. Shew that the equation to the plane containing the line $y/b+z/c=1$, $x=0$; and parallel to the line $x/a-z/c=1$, $y=0$ is $x/a-y/b-z/c+1=0$, and if $2d$ is the S.D. prove that $\dfrac{1}{d^2}=\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}.$

Ex. 15. Two straight lines

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}; \quad \frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}$$

are cut by a third whose direction-cosines are λ, μ, ν. Shew that the length intercepted on the third line is given by

$$\begin{vmatrix} \alpha-\alpha', & \beta-\beta', & \gamma-\gamma' \\ l, & m, & n \\ l', & m', & n' \end{vmatrix} \div \begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ \lambda, & \mu, & \nu \end{vmatrix},$$

and deduce the length of the S.D.

***Ex. 16.** The axes are oblique and the plane ABC has equation $x/a+y/b+z/c=1$. Prove that if the tetrahedron OABC has two pairs of opposite edges at right angles, $\dfrac{\cos\lambda}{a}=\dfrac{\cos\mu}{b}=\dfrac{\cos\nu}{c}$ $(=k)$, and that the equations to the four perpendiculars are

$$\frac{\partial\phi}{\partial y}=2a\cos\nu, \quad \frac{\partial\phi}{\partial z}=2a\cos\mu, \text{ etc.,} \quad \text{and} \quad a\frac{\partial\phi}{\partial x}=b\frac{\partial\phi}{\partial y}=c\frac{\partial\phi}{\partial z}.$$

Hence shew that the perpendiculars pass through the point given by $\dfrac{\partial\phi}{\partial x}=2bck$, $\dfrac{\partial\phi}{\partial y}=2cak$, $\dfrac{\partial\phi}{\partial z}=2abk$. Prove also that the equations to the S.D. of AB and OC are $\dfrac{\partial\phi}{\partial z}=2abk$, $a\dfrac{\partial\phi}{\partial a}=b\dfrac{\partial\phi}{\partial y}$; and that the S.D. passes through the point of concurrence of the perpendiculars.

50. Problems relating to two non-intersecting lines. When two non-intersecting lines are given, the following systems of coordinate axes allow their equations to be written in simple forms, and are therefore of use in problems relating to the lines.

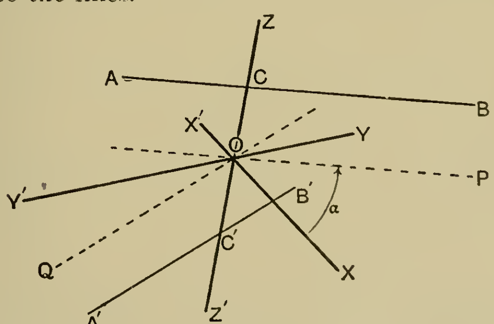

Fig. 25.

I. *Rectangular axes.* Let AB, A′B′, (fig. 25), be the lines, and let CC′, length $2c$, be the shortest distance between them. Take the axis of z along CC′, and O the mid-point

of CC′ as origin. Draw OP, OQ parallel to AB, A′B′, and take the plane POQ as the plane $z=0$. As x- and y-axes take the bisectors of the angles between OP and OQ. Then if the angle between the given lines is 2α, the equations to the planes POZ, QOZ are $y=x\tan\alpha$, $y=-x\tan\alpha$; and hence the equations to AB and A′B′ are

$$y=x\tan\alpha,\ z=c;\quad y=-x\tan\alpha,\ z=-c.$$

These may be written in the symmetrical forms

$$\frac{x}{\cos\alpha}=\frac{y}{\sin\alpha}=\frac{z-c}{0};\quad \frac{x}{\cos\alpha}=\frac{y}{-\sin\alpha}=\frac{z+c}{0}.$$

Ex. 1. P and P′ are variable points on two given non-intersecting lines AB and A′B′, and Q is a variable point so that QP, QP′ are at right angles to one another and at right angles to AB and A′B′ respectively. Find the locus of Q.

Take as the equations to AB, A′B′, $y=mx$, $z=c$; $y=-mx$, $z=-c$. Then the coordinates of P, P′ are α, $m\alpha$, c: β, $-m\beta$, $-c$, where α and β are variables. Let Q be (ξ, η, ζ), then since PQ is perpendicular to AB,

$$(\xi-\alpha)+m(\eta-m\alpha)=0;\ \ldots\ldots(1)$$

since P′Q is perpendicular to A′B′,

$$(\xi-\beta)-m(\eta+m\beta)=0;\ \ldots\ldots(2)$$

since PQ is perpendicular to P′Q,

$$(\xi-\alpha)(\xi-\beta)+(\eta-m\alpha)(\eta+m\beta)+(\zeta-c)(\zeta+c)=0.\ \ldots\ldots(3)$$

To find the equation to the locus we have to eliminate α and β between (1), (2), (3).

The result is easily found to be $\dfrac{m^2\xi^2-\eta^2}{(1+m^2)^2}=\dfrac{\zeta^2-c^2}{1-m^2}$, which represents a hyperboloid.

II. *Axes partly rectangular.* If we take OP and OQ as axes of x and y, instead of the bisectors of the angles between them, we have a system of axes in which the angles ZOX, YOZ are right angles and the angle XOY is the angle between the lines. The equations to AB, A′B′ referred to this system are

$$y=0,\ z=c;\quad x=0,\ z=-c.$$

Ex. 2. P, P′ are variable points on two given non-intersecting lines and PP′ is of constant length $2k$. Find the surface generated by PP′.

Take as the equations to the lines $y=0$, $z=c$; $x=0$, $z=-c$; then P and P′ are $(\alpha, 0, c)$, $(0, \beta, -c)$, where α and β are variables. The equations to PP′ are

$$\frac{x}{\alpha}=\frac{y-\beta}{-\beta}=\frac{z+c}{2c}.\ \ldots\ldots(1)$$

If Q, Q′ are the projections of P, P′ on the plane OXY, PQ = Q′P′ = c, OQ = α, OQ′ = β and QQ′$^2 = \alpha^2 + \beta^2 - 2\alpha\beta\cos\theta$, where θ is the angle between the lines. Therefore

$$PP'^2 = \alpha^2 + \beta^2 - 2\alpha\beta\cos\theta + 4c^2 = 4k^2. \quad \text{.................(2)}$$

To obtain the equation to the locus of PP′ we have to eliminate α and β between the equations (1) and (2). From (1),

$$\alpha = \frac{2cx}{z+c}, \quad \beta = \frac{-2cy}{z-c},$$

and therefore the surface is given by

$$\frac{x^2}{(z+c)^2} + \frac{y^2}{(z-c)^2} + \frac{2xy\cos\theta}{z^2-c^2} = \frac{k^2}{c^2} - 1.$$

Ex. 3. Find the surface generated by a straight line which intersects two given lines and is parallel to a given plane.

If the axes be chosen as in Ex. 2, and the given plane be $lx + my + nz = 0$, the locus is $\dfrac{lx}{z+c} + \dfrac{my}{z-c} + n = 0$.

III. *Axes oblique.* If a point on each of the given lines is specified and a rectangular system is not necessary, the line joining the given points may be taken as z-axis, its mid-point as origin, and the parallels through the origin to the given lines as x- and y-axes. The equations to the lines are then
$$y = 0,\ z = c: \quad x = 0,\ z = -c:$$
where $2c$ is the distance between the given points.

Ex. 4. AP, A′P′ are two given lines, A and A′ being fixed, and P and P′ variable points such that AP . A′P′ is constant. Find the locus of PP′.

Take AA′ as z-axis, etc. Then P, P′ are $(\alpha, 0, c)$, $(0, \beta, -c)$, where $\alpha\beta = \text{constant} = 4k^2$, say. The equations to PP′ are

$$\frac{x}{\alpha} = \frac{y-\beta}{-\beta} = \frac{z+c}{2c},$$

and eliminating α and β between these and $\alpha\beta = 4k^2$, we obtain the equation to the locus, $c^2xy + k^2(z^2 - c^2) = 0$.

Ex. 5. Find the locus of PP′ when (i) AP + A′P′, (ii) AP : A′P′, (iii) AP^2 + A′P′^{2} is constant. Find also the locus of the mid-point of PP′.

Ex. 6. Find the locus of the mid-points of lines whose extremities are on two given lines and which are parallel to a given plane.

Ex. 7. Find the locus of a straight line that intersects two given lines and makes a right angle with one of them.

Ex. 8. Find the locus of a point which is equidistant from two given straight lines.

Ex. 9. Shew that the locus of the mid-points of lines of constant length which have their extremities on two given lines is an ellipse whose centre bisects the S.D., and whose axes are equally inclined to the lines.

Ex. 10. A point moves so that the line joining the feet of the perpendiculars from it to two given lines subtends a right angle at the mid-point of their S.D. Shew that its locus is a hyperbolic cylinder.

Ex. 11. Prove that the locus of a line which meets the lines $y=\pm mx$, $z=\pm c$; and the circle $x^2+y^2=a^2$, $z=0$ is

$$c^2m^2(cy-mxz)^2+c^2(yz-cmx)^2=a^2m^2(z^2-c^2)^2.$$

THE VOLUME OF A TETRAHEDRON.

51. *To find the volume in terms of the coordinates of the vertices, the axes being rectangular.*

If A, B, C are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), the equation to the plane ABC is $\begin{vmatrix} x, & y, & z, & 1 \\ x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \end{vmatrix}=0$, or

$$x\begin{vmatrix} y_1, & z_1, & 1 \\ y_2, & z_2, & 1 \\ y_3, & z_3, & 1 \end{vmatrix}+y\begin{vmatrix} z_1, & x_1, & 1 \\ z_2, & x_2, & 1 \\ z_3, & x_3, & 1 \end{vmatrix}+z\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix}=\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} \quad \text{.....(1)}$$

The equation to the plane ABC can also be written

$$p=x\cos\alpha+y\cos\beta+z\cos\gamma. \quad \text{...............(2)}$$

Let Δ denote the area ABC; then its projections on the planes YOZ, ZOX, XOY are $\cos\alpha.\Delta$, $\cos\beta.\Delta$, $\cos\gamma.\Delta$ respectively. But the projections of A, B, C on the plane YOZ are $(0, y_1, z_1)$, $(0, y_2, z_2)$, $(0, y_3, z_3)$, and therefore the area of the projection of ABC is given in magnitude and sign by $\frac{1}{2}\begin{vmatrix} y_1, & z_1, & 1 \\ y_2, & z_2, & 1 \\ y_3, & z_3, & 1 \end{vmatrix}$. Therefore we have

$$\cos\alpha.\Delta=\tfrac{1}{2}\begin{vmatrix} y_1, & z_1, & 1 \\ y_2, & z_2, & 1 \\ y_3, & z_3, & 1 \end{vmatrix}, \text{ in magnitude and sign.}$$

Hence, using the similar expressions for $\cos\beta.\Delta$ and $\cos\gamma.\Delta$, equation (1) may be written

$$2\Delta(x\cos\alpha+y\cos\beta+z\cos\gamma)=\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}=2p\Delta, \text{ by (2).}$$

Now the absolute measure of $\frac{1}{3}p\Delta$ is the volume of the tetrahedron OABC, and we can introduce positive and negative volume by defining the volume OABC to be $\frac{1}{3}p\Delta$, which is positive or negative according as the direction of rotation determined by ABC is positive or negative for the plane ABC, (p is positive as in § 37). We may then write

$$\text{Vol. OABC}=\text{Vol. OCAB}=\text{Vol. OBCA}=\frac{1}{6}\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix};$$

$$\text{Vol. OBAC}=\frac{1}{6}\begin{vmatrix} x_2, & y_2, & z_2 \\ x_1, & y_1, & z_1 \\ x_3, & y_3, & z_3 \end{vmatrix}=-\text{Vol. OABC, etc.}$$

If D is the point (x_4, y_4, z_4), changing the origin to D, we have

$$\text{Vol. DABC}=\frac{1}{6}\begin{vmatrix} x_1-x_4, & y_1-y_4, & z_1-z_4 \\ x_2-x_4, & y_2-y_4, & z_2-z_4 \\ x_3-x_4, & y_3-y_4, & z_3-z_4 \end{vmatrix}$$

$$=\frac{1}{6}\begin{vmatrix} x_1-x_4, & y_1-y_4, & z_1-z_4, & 0 \\ x_2-x_4, & y_2-y_4, & z_2-z_4, & 0 \\ x_3-x_4, & y_3-y_4, & z_3-z_4, & 0 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix}$$

$$=\frac{1}{6}\begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix}=-\frac{1}{6}\begin{vmatrix} x_4, & y_4, & z_4, & 1 \\ x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \end{vmatrix}.$$

Since the sign of a determinant is changed when two adjacent rows or columns are interchanged, it follows that

$$\text{Vol. DABC}=-\text{Vol. ADBC}=\text{Vol. ABDC}=-\text{Vol. ABCD, etc.}$$

Again, since

$$\begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix} \equiv \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} - \begin{vmatrix} x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \\ x_4, & y_4, & z_4 \end{vmatrix} + \begin{vmatrix} x_3, & y_3, & z_3 \\ x_4, & y_4, & z_4 \\ x_1, & y_1, & z_1 \end{vmatrix} - \begin{vmatrix} x_4, & y_4, & z_4 \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix},$$

$$-\text{Vol. ABCD} = \text{Vol. OABC} - \text{Vol. OBCD} + \text{Vol. OCDA} - \text{Vol. ODAB}.$$

Since the volume ABCD does not depend on the position of the origin, this must be true for all positions of O.

Cor. If α, β, γ, δ are the perpendiculars from any point O to the faces ABC, BCD, CDA, DAB of a given tetrahedron $a\alpha + b\beta + c\gamma + d\delta$ is constant, where a, b, c, d are certain constants.

Ex. 1. A, B, C are (3, 2, 1), (−2, 0, −3), (0, 0, −2). Find the locus of P if the vol. PABC $=5$. *Ans.* $2x+3y-4z=38$.

Ex. 2. The lengths of two opposite edges of a tetrahedron are a, b, their S.D. is equal to d, and the angle between them to θ; prove that the volume is $\dfrac{abd \sin\theta}{6}$.

Ex. 3. AA′ is the S.D. between two given lines, and B, B′ are variable points on them such that the volume AA′BB′ is constant. Prove that the locus of the mid-point of BB′ is a hyperbola whose asymptotes are parallel to the lines.

Ex. 4. If O, A, B, C, D are any five points, and p_1, p_2, p_3, p_4 are the projections of OA, OB, OC, OD on any given line, prove that

$$p_1 \,.\, \text{Vol. OBCD} - p_2 \,.\, \text{Vol. OCDA} + p_3 \,.\, \text{Vol. ODAB} - p_4 \,.\, \text{Vol. OABC} = 0.$$

Ex. 5. Prove that the volume of a tetrahedron, two of whose sides are of constant length and lie upon given straight lines, is constant, and that the locus of its centre of gravity is a straight line.

Ex. 6. If A, B, C, D are coplanar and A′, B′, C′, D′ are their projections on any plane, prove that Vol. AB′C′D′ $=-$Vol. A′BCD.

Ex. 7. Lines are drawn in a given direction through the vertices A, B, C, D of a tetrahedron to meet the opposite faces in A′, B′, C′, D′. Prove that Vol. A′B′C′D′ $=-3$. Vol. ABCD.

***Ex. 8.** Find the volume of the tetrahedron the equations to whose faces are $a_r x + b_r y + c_r z + d_r = 0$, $r = 1, 2, 3, 4$.

Let the planes corresponding to $r=1, 2, 3, 4$ be BCD, CDA, DAB, ABC respectively, and let $\Delta \equiv \begin{vmatrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \\ a_4, & b_4, & c_4, & d_4 \end{vmatrix}$. Then (x_1, y_1, z_1),

the coordinates of A_1 are given by

$$\frac{x_1}{A_1}=\frac{y_1}{B_1}=\frac{z_1}{C_1}=\frac{1}{D_1}, \text{ where } A_1=\frac{\partial\Delta}{\partial a_1}, \text{ etc.};$$

and therefore the volume is given by

$$-\frac{1}{6}\begin{vmatrix} A_1/D_1, & B_1/D_1, & C_1/D_1, & 1 \\ A_2/D_2, & & \text{etc.} & \end{vmatrix} = -\frac{1}{6D_1D_2D_3D_4}\begin{vmatrix} A_1, & B_1, & C_1, & D_1 \\ A_2, & & \text{etc.} & \end{vmatrix}$$

$$=\frac{-\Delta^3}{6D_1D_2D_3D_4}.$$

(C. Smith, *Algebra*, p. 544.)

*__Ex. 9.__ The lengths of the edges OA, OB, OC of a tetrahedron OABC are a, b, c, and the angles BOC, COA, AOB are λ, μ, ν; find the volume.

Suppose that the direction-cosines of OA, OB, OC, referred to rectangular axes through O, are l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3; then the coordinates of A are l_1a, m_1a, n_1a, etc.

Therefore $$6\,.\,\text{Vol. OABC}=\begin{vmatrix} l_1a, & m_1a, & n_1a \\ l_2b, & m_2b, & n_2b \\ l_3c, & m_3c, & n_3c \end{vmatrix}$$

$$=\pm abc\begin{vmatrix} \Sigma l_1^2, & \Sigma l_1l_2, & \Sigma l_1l_3 \\ \Sigma l_1l_2, & \Sigma l_2^2, & \Sigma l_2l_3 \\ \Sigma l_1l_3, & \Sigma l_2l_3, & \Sigma l_3^2 \end{vmatrix}^{\frac{1}{2}}=\pm abc\begin{vmatrix} 1, & \cos\nu, & \cos\mu \\ \cos\nu, & 1, & \cos\lambda \\ \cos\mu, & \cos\lambda, & 1 \end{vmatrix}^{\frac{1}{2}}.$$

(Cf. § 27, Ex. 3.)

CHAPTER IV.

CHANGE OF AXES.

52. OX, OY, OZ ; Oξ, Oη, Oζ are two sets of rectangular axes through a common origin O, and the direction-cosines of Oξ, Oη, Oζ, referred to OX, OY, OZ, are l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3. P, any point, has coordinates x, y, z referred to OX, OY, OZ and ξ, η, ζ referred to Oξ, Oη, Oζ. We have to

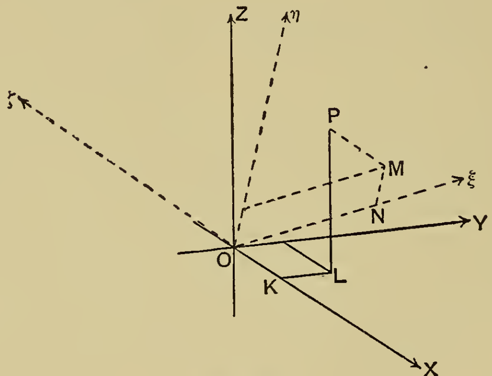

FIG. 26.

express x, y, z in terms of ξ, η, ζ and the direction-cosines, and *vice-versa*.

In the accompanying figure, ON, NM, MP represent ξ, η, ζ, and OK, KL, LP represent x, y, z. Projecting OP and ON, NM, MP on OX, OY, OZ in turn, we obtain

$$\left.\begin{aligned} x &= l_1\xi + l_2\eta + l_3\zeta, \\ y &= m_1\xi + m_2\eta + m_3\zeta, \\ z &= n_1\xi + n_2\eta + n_3\zeta. \end{aligned}\right\} \quad \dots\dots\dots\dots\dots(1)$$

And projecting OP and OK, KL, LP on Oξ, Oη, Oζ in turn, we obtain

$$\left.\begin{aligned}\xi &= l_1x + m_1y + n_1z,\\ \eta &= l_2x + m_2y + n_2z,\\ \zeta &= l_3x + m_3y + n_3z.\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

	x	y	z
ξ	l_1	m_1	n_1
η	l_2	m_2	n_2
ζ	l_3	m_3	n_3

The equations (1) and (2) can be derived from the above scheme, which may be constructed as follows: Affix to the columns and rows the numbers x, y, z; ξ, η, ζ; and in the square common to the column headed x and the row headed ξ, place the cosine of the angle between OX and Oξ, *i.e.* l_1, and so on. To obtain the value of x, multiply the numbers in the x-column by the numbers at the left of their respective rows and add the products; to obtain the value of ξ, multiply the numbers in the ξ-row by the numbers at the heads of their respective columns, and add the products. Similarly, any other of the equations (1) and (2) may be derived.

Cor. Since x, y, z are linear functions of ξ. η, ζ, the degree of an equation is unaltered by transformation from any one set of rectangular axes to any other. For it is evident that the degree cannot be raised. Neither can it be lowered, since in changing again to the original axes, it would require to be raised.

53. Relations between the direction-cosines of three mutually perpendicular lines. We have

$$\left.\begin{aligned}l_1^2+m_1^2+n_1^2&=1,\\ l_2^2+m_2^2+n_2^2&=1,\\ l_3^2+m_3^2+n_3^2&=1,\end{aligned}\right\} \ldots\ldots(\text{A}) \qquad \left.\begin{aligned}l_2l_3+m_2m_3+n_2n_3&=0,\\ l_3l_1+m_3m_1+n_3n_1&=0,\\ l_1l_2+m_1m_2+n_1n_2&=0.\end{aligned}\right\} \ldots\ldots(\text{B})$$

From the second and third equations of (B), we derive

$$\frac{l_1}{m_2n_3-m_3n_2}=\frac{m_1}{n_2l_3-n_3l_2}=\frac{n_1}{l_2m_3-l_3m_2};$$

$$\text{and each}=\frac{l_1^2+m_1^2+n_1^2}{\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}}=\frac{\pm\sqrt{l_1^2+m_1^2+n_1^2}}{\sqrt{\Sigma(m_2n_3-m_3n_2)^2}}=\pm 1.$$

(§ 23, *Cor.* 1.)

Therefore, if $D^{-1}\equiv\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}=\pm 1,$

$$\left.\begin{array}{l} l_1=D(m_2n_3-m_3n_2),\ m_1=D(n_2l_3-n_3l_2),\ n_1=D(l_2m_3-l_3m_2). \\ \text{Similarly,} \\ l_2=D(m_3n_1-m_1n_3),\ m_2=D(n_3l_1-n_1l_3),\ n_2=D(l_3m_1-l_1m_3); \\ l_3=D(m_1n_2-m_2n_1),\ m_3=D(n_1l_2-n_2l_1),\ n_3=D(l_1m_2-l_2m_1). \end{array}\right\}\text{(E)}$$

Multiplying the first column of equations (E) by l_1, l_2, l_3 respectively, and adding, we obtain

$$l_1^2+l_2^2+l_3^2=D\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}=1;$$

and similarly,

$$m_1^2+m_2^2+m_3^2=1,\quad n_1^2+n_2^2+n_3^2=1.\ \ldots\ldots\ldots\text{(C)}$$

Multiplying the second column by n_1, n_2, n_3, we obtain in the same way,

$$m_1n_1+m_2n_2+m_3n_3=0,$$

and similarly,

$$n_1l_1+n_2l_2+n_3l_3=0,$$

$$l_1m_1+l_2m_2+l_3m_3=0.\ \ldots\ldots\ldots\ldots\ldots\text{(D)}$$

The equations (C) and (D) can be derived at once from the consideration that $l_1, l_2, l_3 : m_1, m_2, m_3 ; n_1, n_2, n_3$ are the direction-cosines of OX, OY, OZ referred to $O\xi$, $O\eta$, $O\zeta$. The method adopted shews that the four sets (A), (B), (C), (D) are not independent, and it can be shewn as above, that if either of the two dissimilar sets (A), (B); (C), (D) be given, the other two can be deduced.

Suppose that a plane LMN (fig. 27) cuts off three positive segments of unit length from the axes $O\xi$, $O\eta$, $O\zeta$. Then if

the direction of rotation given by LMN is the positive direction of rotation for the plane LMN, the system of axes Oξ, Oη, Oζ can be brought by rotation about O into coincidence with the system OX, OY, OZ. If the direction of rotation is negative, and Oξ, Oη are brought to coincide with OX, OY respectively, then Oζ coincides with OZ$'$.

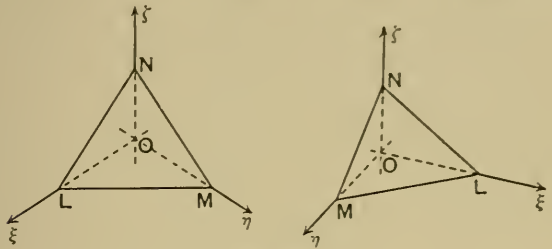

FIG. 27.

Now $\quad 6 \,.\, \text{Vol. OLMN} = \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} \equiv D^{-1}.$

The volume, and therefore D, is positive or negative according as the direction of rotation determined by LMN is positive or negative for the plane LMN. Hence if LMN gives the positive direction of rotation, from equations (E),

$$l_1 = +(m_2 n_3 - m_3 n_2), \text{ etc.,}$$

the positive sign being taken throughout. If LMN gives the negative direction of rotation,

$$l_1 = -(m_2 n_3 - m_3 n_2), \text{ etc.,}$$

the negative sign being taken throughout.

Conversely, if l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 are the direction-cosines of three mutually perpendicular directed lines Oξ, Oη, Oζ, and

$$l_1 = +(m_2 n_3 - m_3 n_2),$$

then Oξ, Oη, Oζ can be brought by rotation about O to coincide with OX, OY, OZ.

Ex. Verify the above results by considering Oξ, Oη, Oζ to coincide, say, with OX$'$, OY$'$, OZ.

Here $l_1 = -1$, $m_1 = n_1 = 0$; $l_2 = n_2 = 0$, $m_2 = -1$; $l_3 = m_3 = 0$, $n_3 = 1$. $l_1 = m_2 n_3 - m_3 n_2$, and if Oξ be rotated to coincide with OX, Oη coincides with OY.

54. Section of a surface by a given plane. The following method of transformation can be applied with advantage when the section of a given surface by a given plane passing through the origin is to be considered.

Let the equation to the plane be $lx+my+nz=0$, where $l^2+m^2+n^2=1$, and n is positive.

Take as $O\zeta$, the new axis of z, the normal to the plane which passes through O and makes an acute angle with OZ. Then the equations to $O\zeta$, referred to OX, OY, OZ, are $x/l=y/m=z/n$. Take as $O\eta$, the new y-axis, the line in the plane $ZO\zeta$ which is at right angles to $O\zeta$ and makes an acute angle with OZ. Then choose $O\xi$, the new x-axis, at right angles to $O\eta$ and $O\zeta$, and so that the system $O\xi$, $O\eta$, $O\zeta$ can be brought to coincidence with OX, OY, OZ. The given

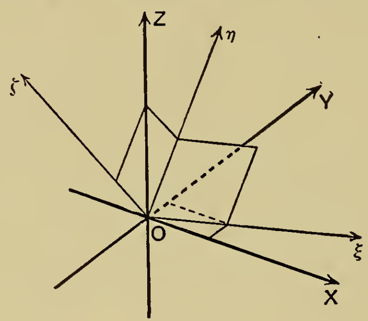

FIG. 28.

plane is $\xi O\eta$, and since $O\xi$ is at right angles to $O\zeta$ and $O\eta$, it is at right angles to OZ which lies in the plane $\zeta O\eta$. Hence $O\xi$ lies in the plane XOY, and therefore is the line of intersection of the given plane and the plane XOY. The equation to the plane $\zeta O\eta$ is $x/l=y/m$; therefore if λ, μ, ν are the direction-cosines of $O\eta$,

$$l\lambda+m\mu+n\nu=0,$$

$$m\lambda-l\mu=0,$$

whence $$\frac{\lambda}{l}=\frac{\mu}{m}=\frac{\nu}{\dfrac{l^2+m^2}{-n}}=\frac{\pm 1}{\dfrac{\sqrt{l^2+m^2}}{n}}.$$

But $O\eta$ makes an acute angle with OZ, and therefore ν is positive, and therefore the negative sign must be taken in the ambiguity.

$$\therefore\ \lambda = \frac{-ln}{\sqrt{l^2+m^2}},\quad \mu = \frac{-mn}{\sqrt{l^2+m^2}},\quad \nu = \sqrt{l^2+m^2}.$$

And since $O\xi$ is at right angles to $O\eta$ and $O\zeta$, by § 53 (E), the direction-cosines of $O\xi$ are

$$n\mu - m\nu,\quad l\nu - n\lambda,\quad m\lambda - l\mu;$$

$$i.e.\ \frac{-m}{\sqrt{l^2+m^2}},\quad \frac{l}{\sqrt{l^2+m^2}},\quad 0.$$

Hence we have the scheme:

	x	y	z
ξ	$\frac{-m}{\sqrt{l^2+m^2}}$	$\frac{l}{\sqrt{l^2+m^2}}$	0
η	$\frac{-ln}{\sqrt{l^2+m^2}}$	$\frac{-mn}{\sqrt{l^2+m^2}}$	$\sqrt{l^2+m^2}$
ζ	l	m	n

Ex. 1. Shew that the projection of a conic is a conic of the same species.

The equation $f(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a cylinder whose generators are parallel to OZ and pass through the conic $z=0$, $f(x, y)=0$. The equations $lx+my+nz=0$, $f(x, y)=0$ represent the curve in which the plane $lx+my+nz=0$ cuts the cylinder, and the projection of this curve on the plane $z=0$ is the conic. Change the axes as above, and the equations to the curve become

$$\zeta=0,\ a\left(\frac{m\xi+ln\eta}{\sqrt{l^2+m^2}}\right)^2 - 2h\left(\frac{m\xi+ln\eta}{\sqrt{l^2+m^2}}\right)\left(\frac{l\xi-mn\eta}{\sqrt{l^2+m^2}}\right) + b\left(\frac{l\xi-mn\eta}{\sqrt{l^2+m^2}}\right)^2 + \ldots = 0,$$

and therefore the curve is the conic given by

$$\zeta=0,\ a'\xi^2 + 2h'\xi\eta + b'\eta^2 + \ldots = 0,$$

where $$a' \equiv \frac{am^2 - 2hlm + bl^2}{l^2+m^2},\quad h' \equiv \frac{lmn(a-b) - hn(l^2-m^2)}{l^2+m^2},$$

$$b' \equiv \frac{n^2(al^2 + 2hlm + bm^2)}{l^2+m^2}.$$

Whence $$h'^2 - a'b' = n^2(h^2 - ab)$$

and $$h'^2 - a'b' \gtreqless 0 \text{ as } h^2 - ab \gtreqless 0.$$

Ex. 2. All plane sections of a surface represented by an equation of the second degree are conics.

Take coordinate axes so that a plane section is $z=0$; the equation to the surface is, after transformation, of the form

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0.$$

The section by the plane XOY is the conic whose equations are

$$z=0, \quad ax^2+2hxy+by^2+2ux+2vy+d=0.$$

The surfaces represented by equations of the second degree are **the conicoids.**

Ex. 3. All parallel plane sections of a conicoid are similar and similarly situated conics.

Take the coordinate plane $z=0$ parallel to a system of parallel plane sections. The equations to the sections by the planes $z=k$, $z=k'$ are then,

$$z=k, \quad ax^2+2hxy+by^2+2x(gk+u)+2y(fk+v)+ck^2+2wk+d=0,$$

$$z=k', \quad ax^2+2hxy+by^2+2x(gk'+u)+2y(fk'+v)+ck'^2+2wk'+d=0.$$

Hence the sections are similar and similarly situated conics.

Ex. 4. Find the conditions that the section of the surface $ax^2+by^2+cz^2=1$ by the plane $lx+my+nz=p$ should be (i) a parabola, (ii) an ellipse, (iii) a hyperbola.

(It is sufficient to examine the section by the plane $lx+my+nz=0$, which, by Ex. 3, is a similar conic. The equation to the projection of this section on the plane $z=0$ is obtained by eliminating z between the equations $lx+my+nz=0$, $ax^2+by^2+cz^2=1$, and the projection is a conic of the same species.)

Ans. For a parabola $l^2/a+m^2/b+n^2/c=0$, etc.

Ex. 5. Find the condition that the section of $ax^2+by^2=2z$ by $lx+my+nz=p$ should be a rectangular hyperbola.

(Since rectangular hyperbolas do not, in general, project into rectangular hyperbolas, it will, in this case, be necessary to examine the actual section of the surface by the plane $lx+my+nz=0$ by the method of § 54.) *Ans.* $a+b-al^2-bm^2=0$.

Ex. 6. Find the conditions that the section of $ax^2+by^2+cz^2=1$ by $lx+my+nz=p$ should be a circle.

Ans. $l=0$, $m^2(c-a)=n^2(a-b)$; or $m=0$, $n^2(a-b)=l^2(b-c)$; or $n=0$, $l^2(b-c)=m^2(c-a)$.

Ex. 7. If $lx+my=0$ is a circular section of

$$\mathsf{A}x^2+\mathsf{B}y^2+\mathsf{C}z^2+2\mathsf{D}xy=1,$$

prove that $$(\mathsf{B}-\mathsf{C})l^2-2\mathsf{D}lm+(\mathsf{A}-\mathsf{C})m^2=0.$$

Ex. 8. Prove that the eccentricity of the section of $xy=z$ by $lx+my+nz=0$, $(l^2+m^2+n^2=1)$, is given by

$$\frac{2}{e^2}=1\pm\frac{lm}{\sqrt{(n^2+l^2)(n^2+m^2)}}.$$

Explain the result when $n=0$.

Ex. 9. Shew that if

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d$$

be transformed by change of coordinates from one set of rectangular axes to another with the same origin, the expressions $a+b+c$, $u^2+v^2+w^2$ remain unaltered in value.

Ex. 10. Two sets of rectangular axes through a common origin O meet a sphere whose centre is O in P, Q, R; P′, Q′, R′. Prove that Vol. OPQR′ = ± Vol. OP′Q′R.

Ex. 11. The equations, referred to rectangular axes, of three mutually perpendicular planes, are $p_r - l_r x - m_r y - n_r z = 0$, $r=1, 2, 3$. Prove that if (ξ, η, ζ) is at a distance d from each of them,

$$\frac{\xi-(l_1p_1+l_2p_2+l_3p_3)}{l_1+l_2+l_3}=\frac{\eta-(m_1p_1+m_2p_2+m_3p_3)}{m_1+m_2+m_3}$$
$$=\frac{\zeta-(n_1p_1+n_2p_2+n_3p_3)}{n_1+n_2+n_3}=d.$$

Ex. 12. If the axes of x, y, z are rectangular, prove that the substitutions

$$x=\frac{\xi}{\sqrt{3}}+\frac{\eta}{\sqrt{2}}+\frac{\zeta}{\sqrt{6}},\quad y=\frac{\xi}{\sqrt{3}}-\frac{2\zeta}{\sqrt{6}},\quad z=\frac{\xi}{\sqrt{3}}-\frac{\eta}{\sqrt{2}}+\frac{\zeta}{\sqrt{6}}$$

give a transformation to another set of rectangular axes in which the plane $x+y+z=0$ becomes the plane $\xi=0$, and hence prove that the section of the surface $yz+zx+xy+a^2=0$ by the plane $x+y+z=0$ is a circle of radius $\sqrt{2}\,.\,a$.

***55.** If OX, OY, OZ are rectangular axes, and Oξ, Oη, Oζ are oblique axes whose direction-cosines, referred to OX, OY, OZ, are l_1, m_1, n_1; l_2, m_2, n_2, l_3, m_3, n_3, then projecting on OX, OY, OZ; Oξ, Oη, Oζ, as in § 52, we obtain

$$\left.\begin{aligned} x &= l_1\xi+l_2\eta+l_3\zeta,\\ y &= m_1\xi+m_2\eta+m_3\zeta,\\ z &= n_1\xi+n_2\eta+n_3\zeta.\end{aligned}\right\}\quad\text{.....................(A)}$$

$$\left.\begin{aligned} \xi+\eta\cos\nu+\zeta\cos\mu &= l_1x+m_1y+n_1z,\\ \xi\cos\nu+\eta+\zeta\cos\lambda &= l_2x+m_2y+n_2z,\\ \xi\cos\mu+\eta\cos\lambda+\zeta &= l_3x+m_3y+n_3z,\end{aligned}\right\}\quad\text{...........(B)}$$

where the angles ηOζ, ζOξ, ξOη are λ, μ, ν. The equations (B) can also be deduced from (A) by multiplying in turn by l_1, m_1, n_1, etc., and adding. Again, from (A),

$$\xi=\begin{vmatrix} x, & y, & z\\ l_2, & m_2, & n_2\\ l_3, & m_3, & n_3\end{vmatrix}\div\begin{vmatrix} l_1, & m_1, & n_1\\ l_2, & m_2, & n_2\\ l_3, & m_3, & n_3\end{vmatrix},\text{ etc., etc. ...(C)}$$

By means of (A) and (C) we can transform from rectangular to oblique axes and *vice versa*.

Cor. Since x, y, z are linear functions of ξ, η, ζ and *vice versa*, the degree of any equation is unaltered by transformation from rectangular to oblique axes or from oblique to rectangular axes. The transformation from one set of oblique axes to another can be performed, by introducing a set of rectangular axes, in the above two steps, and hence in this most general case the degree of the equation is unaltered by the transformation.

Ex. 1. The equation $x^2+4(y^2+z^2)=2$ is transformed by change from rectangular axes, the new axes being oblique, and having direction-cosines proportional to

$$2, 1, 1;\quad 4, \sqrt{3}-1, -\sqrt{3}-1;\quad 4, -\sqrt{3}-1, \sqrt{3}-1.$$

Shew that the new equation is $x^2+y^2+z^2=1$.

Ex. 2. If P, Q, R are (ξ_r, η_r, ζ_r), $r=1, 2, 3$, referred to a set of oblique axes through an origin O, prove that

$$6 \cdot \text{Vol. OPQR} = \begin{vmatrix} \xi_1, & \eta_1, & \zeta_1 \\ \xi_2, & \eta_2, & \zeta_2 \\ \xi_3, & \eta_3, & \zeta_3 \end{vmatrix} \cdot \begin{vmatrix} 1, & \cos\nu, & \cos\mu \\ \cos\nu, & 1, & \cos\lambda \\ \cos\mu, & \cos\lambda, & 1 \end{vmatrix}^{\frac{1}{2}}.$$

(Use § 55 (B); cf. § 51, Ex. 9.)

*Examples I.

1. The gnomon of a sundial is in the meridian at an elevation λ (equal to the latitude), and the sun is due east at an elevation α. Find the angle θ that the shadow makes with the N. and S. line of the dial.

2. Find the equations to the line through (1, 1, 1) which meets both the lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-2}{4}$, $x=2y=3z$, and shew that its intersection with the second line is $\left(\frac{15}{26}, \frac{15}{52}, \frac{5}{26}\right)$.

3. If OA, OB, OC have direction-ratios l_r, m_r, n_r, $r=1, 2, 3$; and OA′, OB′, OC′ bisect the angles BOC, COA, AOB, the planes AOA′, BOB′, COC′ pass through the line

$$\frac{x}{l_1+l_2+l_3}=\frac{y}{m_1+m_2+m_3}=\frac{z}{n_1+n_2+n_3}.$$

4. P is a given point and PM, PN are the perpendiculars from P to the planes ZOX, XOY. OP makes angles θ, α, β, γ with the planes OMN and the (rectangular) coordinate planes. Prove that

$$\operatorname{cosec}^2\theta=\operatorname{cosec}^2\alpha+\operatorname{cosec}^2\beta+\operatorname{cosec}^2\gamma.$$

5. Shew that the locus of lines which meet the lines

$$\frac{x+a}{0}=\frac{y}{\sin\alpha}=\frac{z}{\pm\cos\alpha}$$

at the same angle is

$$(xy\cos\alpha-az\sin\alpha)(xz\sin\alpha-ay\cos\alpha)=0.$$

6. Find the locus of a straight line which meets OX and the circle $x^2+y^2=c^2$, $z=h$, so that the distance between the points of section is $\sqrt{c^2+h^2}$.

7. If three rectangular axes be rotated about the line $\frac{x}{\lambda}=\frac{y}{\mu}=\frac{z}{\nu}$ into new positions, and the direction-cosines of the new axes referred to the old are l_1, m_1, n_1, etc.; then if

$$l_1=+(m_2n_3-m_3n_2),\quad \lambda(m_3+n_2)=\mu(n_1+l_3)=\nu(l_2+m_1);$$

also if θ is the angle through which the system is rotated,

$$\sin^2\frac{\theta}{2}=\frac{1-l_1}{2}\cdot\frac{\lambda^2+\mu^2+\nu^2}{\mu^2+\nu^2}.$$

8. If the shortest distances between lines 1, 2, 3 are parallel to lines 4, 5, 6, then the shortest distances between the lines 4, 5, 6 are parallel to the lines 1, 2, 3.

9. Any three non-intersecting lines can be made the edges of a parallelepiped, and if the lines are $\frac{x-\alpha_r}{l_r}=\frac{y-\beta_r}{m_r}=\frac{z-\gamma_r}{n_r}$, $r=1, 2, 3$, the lengths of the edges are

$$\begin{vmatrix} \alpha_2-\alpha_3, & \beta_2-\beta_3, & \gamma_2-\gamma_3 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} \div \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}, \text{ etc.}$$

Consider the case where the denominator is zero.

10. OA, OB, OC are edges of a parallelepiped and R is the corner opposite to O. OP and RQ are perpendiculars to the plane ABC. Compare the lengths of OP and RQ. If the figure is rectangular and O is taken as origin, and the plane ABC is given by $lx+my+nz=p$, PQ has direction-cosines proportional to $l^{-1}-3l$, $m^{-1}-3m$, $n^{-1}-3n$, and $PQ^2=OR^2-9\,.\,OP^2$.

11. OS is the diagonal of the cube of which OP, OQ, OR are edges. OU is the diagonal of the parallelepiped of which OQ, OR, OS are edges, and OV and OW are formed similarly. Find the coordinates of U, V, W, and if OT is the diagonal of the parallelepiped of which OU, OV, OW are edges, shew that OT coincides with OS and that $OT=5\,.\,OS$.

12. Find the equations to the straight line through the origin which meets at right angles the line whose equations are

$$(b+c)x+(c+a)y+(a+b)z=k=(b-c)x+(c-a)y+(a-b)z,$$

and find the coordinates of the points of section.

13. Find the locus of a point which moves so that the ratio of its distances from two given lines is constant.

14. A line is parallel to the plane $y+z=0$ and intersects the circles $x^2+y^2=a^2$, $z=0$; $x^2+z^2=a^2$, $y=0$; find the surface it generates.

15. Find the equation to the surface generated by a straight line which is parallel to the line $y=mx$, $z=nx$, and intersects the ellipse

$$x^2/a^2+y^2/b^2=1, \quad z=0.$$

16. A plane triangle, sides a, b, c, is placed so that the mid-points of the sides are on the axes (rectangular). Shew that the lengths intercepted on the axes are given by

$$l^2=(b^2+c^2-a^2)/8, \quad m^2=(c^2+a^2-b^2)/8, \quad n^2=(a^2+b^2-c^2)/8,$$

and that the coordinates of the vertices are $(-l, m, n)$, $(l, -m, n)$, $(l, m, -n)$.

17. Lines are drawn to meet two given lines and touch the right circular cylinder whose axis is the S.D. (length $2c$), and radius c. Find the surface generated.

18. The section of $ax^2+by^2+cz^2=1$ by the plane $lx+my+nz=p$ is a parabola of latus rectum $2L$. Prove that

$$L(l^2/a^2+m^2/b^2+n^2/c^2)^{\frac{3}{2}}=p(l^2+m^2+n^2)/abc.$$

19. A line moves so as to intersect the line $z=0$, $x=y$; and the circles $x=0$, $y^2+z^2=r^2$; $y=0$, $z^2+x^2=r^2$. Prove that the equation to the locus is

$$(x+y)^2\{z^2+(x-y)^2\}=r^2(x-y)^2.$$

20. Prove that $\dfrac{a}{y-z}+\dfrac{b}{z-x}+\dfrac{c}{x-y}=0$ represents a pair of planes whose line of intersection is equally inclined to the axes.

21. Find the surface generated by a straight line which revolves about a given straight line at a constant distance from it and makes a given angle with it.

22. Shew that $x^2+y^2+z^2-3xy-3zx-3yz=1$ represents a surface of revolution about the line $x=y=z$, and find the equations to the generating curve.

23. L_1, L_2, L_3 are three given straight lines and the directions of L_1 and L_2 are at right angles. Find the locus of the line joining the feet of the perpendiculars from any point on L_3 to L_1 and L_2.

24. The ends of diameters of the ellipse $z=c$, $x^2/a^2+y^2/b^2=1$ are joined to the corresponding ends of the conjugates of parallel diameters of the ellipse $x^2/a^2+y^2/b^2=1$, $z=-c$. Find the equation to the surface generated by the joining lines.

25. A and B are two points on a given plane and AP, BQ are two lines in given directions at right angles to AB. Shew that for all lines PQ, parallel to the plane, AP : BQ is constant, and that all such lines lie on a conicoid.

26. The vertex A of a triangle ABC lies on a given line; AB and AC pass through given points; B and C lie on given planes; shew that the locus of BC is a conicoid.

27. Prove that the equation to the two planes inclined at an angle α to the xy-plane and containing the line $y=0$, $z\cos\beta=x\sin\beta$, is

$$(x^2+y^2)\tan^2\beta+z^2-2zx\tan\beta=y^2\tan^2\alpha.$$

28. A line moves so as to meet the lines $\dfrac{x}{\cos\alpha}=\dfrac{y}{\pm\sin\alpha}=\dfrac{z+c}{0}$ in A and B and pass through the curve $yz=k^2$, $x=0$. Prove that the locus of the mid-point of AB is a curve of the third degree, two of whose asymptotes are parallel to the given lines.

29. Given two non-intersecting lines whose directions are at right angles and whose S.D. is AB, and a circle whose centre C is on AB and plane parallel to the lines. Shew that the locus of a variable line which intersects the given lines and circle is a surface whose sections by planes parallel to the lines are ellipses whose centres lie on AB, and that the section by the plane through C′, another point of AB, is a circle, if C, C′ are harmonic conjugates with respect to A and B.

30. If the axes are rectangular the locus of the centre of a circle of radius a which always intersects them is

$$x\sqrt{a^2-y^2-z^2}+y\sqrt{a^2-z^2-x^2}+z\sqrt{a^2-x^2-y^2}=a^2.$$

31. A line is drawn to meet $y=x\tan\alpha$, $z=c$; $y=-x\tan\alpha$, $z=-c$, so that the length intercepted on it is constant. Shew that its equations may be written in the form

$$\frac{x-k\sin\theta\cot\alpha}{k\cos\theta}=\frac{y-k\cos\theta\tan\alpha}{k\sin\theta}=\frac{z}{c},$$

where k is a constant and θ a parameter. Deduce the equation to the locus of the line.

32. Find the equation to the surface generated by a straight line which is parallel to the plane $z=0$ and intersects the line $x=y=z$, and the curve $x+2y=4z$, $x^2+y^2=a^2$.

33. Through a fixed line L, which lies in the xy-plane but does not pass through the origin, is drawn a plane which intersects the planes $x=0$ and $y=0$ in lines M and N respectively. Through M and a fixed point A, and through N and another fixed point B, planes are drawn. Find the locus of their line of intersection.

34. The axes are rectangular and a point P moves on the fixed plane $x/a+y/b+z/c=1$. The plane through P perpendicular to OP meets the axes in A, B, C. The planes through A, B, C parallel to YOZ, ZOX, XOY intersect in Q. Shew that the locus of Q is $\dfrac{1}{x^2}+\dfrac{1}{y^2}+\dfrac{1}{z^2}=\dfrac{1}{ax}+\dfrac{1}{by}+\dfrac{1}{cz}$.

35. AB and CD are given non-intersecting lines. Any plane through AB cuts CD in P, and PQ is normal to it at P. Find the locus of PQ.

36. Find the equation to a plane which touches each of the circles $x=0$, $y^2+z^2=a^2$; $y=0$, $z^2+x^2=b^2$; $z=0$, $x^2+y^2=c^2$. How many such planes are there?

37. Find the locus of the position of the eye at which two given non-intersecting lines appear to cut at right angles.

38. Four given points of a variable line lie on the faces of a quadrilateral prism. Shew that any other point of the line describes a line which is parallel to the edges of the prism.

39. The locus of the harmonic conjugates of P with respect to the two points in which any secant through P cuts a pair of planes is the polar of P with respect to the planes. Prove that the equation to the polar of (x_1, y_1, z_1) with respect to $u=0$, $v=0$, is $\frac{u}{u_1}+\frac{v}{v_1}=0$, where u_1 is the result of substituting x_1, y_1, z_1 for x, y, z in u, etc. Shew also that the polars of P with respect to the pairs of planes that form a trihedral angle cut those planes in three coplanar lines.

40. Any line meets the faces **BCD**, **CDA**, **DAB**, **ABC** of a tetrahedron **ABCD** in **A′**, **B′**, **C′**, **D′**. Prove that the mid-points of **AA′**, **BB′**, **CC′**, **DD′** are coplanar.

41. If the axes are rectangular, and λ, μ, ν are the angles between the lines of intersection of the planes $a_rx+b_ry+c_rz=0$, $r=1, 2, 3$, prove that

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}$$
$$=\frac{(\Sigma a_1^2 . \Sigma a_2^2 . \Sigma a_3^2)^{\frac{1}{2}}(1-\cos^2\lambda-\cos^2\mu-\cos^2\nu+2\cos\lambda\cos\mu\cos\nu)}{\sin\lambda\sin\mu\sin\nu}.$$

42. The equations $x=\lambda z+\mu$, $y=(\lambda^3-2\lambda\mu)z+\mu(\lambda^2-\mu)$, where λ and μ are parameters, determine a system of lines. Find the locus of those which intersect the z-axis. Prove that two lines of the system pass through any given point unless the given point lies on a certain curve, when an infinite number of lines pass through it, and find the equations to the curve.

CHAPTER V.

THE SPHERE.

56. Equation to a sphere. If the axes are rectangular the square of the distance between the points P, (x_1, y_1, z_1) and Q, (x_2, y_2, z_2) is given by $(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2$, and therefore the equation to the sphere whose centre is P and whose radius is of length r, is

$$(x-x_1)^2+(y-y_1)^2+(z-z_1)^2=r^2.$$

Any equation of the form

$$ax^2+ay^2+az^2+2ux+2vy+2wz+d=0$$

can be written

$$\left(x+\frac{u}{a}\right)^2+\left(y+\frac{v}{a}\right)^2+\left(z+\frac{w}{a}\right)^2=\frac{u^2+v^2+w^2-ad}{a^2},$$

and therefore represents a sphere whose centre is

$$\left(-\frac{u}{a},\ -\frac{v}{a},\ -\frac{w}{a}\right) \text{ and radius } \frac{\sqrt{u^2+v^2+w^2-ad}}{a}.$$

Ex. 1. Find the equation to the sphere whose centre is $(2, -3, 4)$ and radius 5. *Ans.* $x^2+y^2+z^2-4x+6y-8z+4=0$.

Ex. 2. Find the centre and radius of the sphere given by

$x^2+y^2+z^2-2x+4y-6z=11$. *Ans.* $(1, -2, 3)$, 5.

Ex. 3. Shew that the equation

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$

represents the sphere on the join of (x_1, y_1, z_1), (x_2, y_2, z_2) as diameter.

Ex. 4. Find the equation to the sphere through the points

$$(0,\ 0,\ 0),\quad (0,\ 1,\ -1),\quad (-1,\ 2,\ 0),\quad (1,\ 2,\ 3).$$

Ans. $7(x^2+y^2+z^2)-15x-25y-11z=0$.

Ex. 5. Find the equation to the sphere which passes through the point (α, β, γ) and the circle $z=0$, $x^2+y^2=a^2$.

Ans. $\gamma(x^2+y^2+z^2-a^2)=z(\alpha^2+\beta^2+\gamma^2-a^2)$.

Ex. 6. Find the equations to the spheres through the circle

$$x^2+y^2+z^2=9, \quad 2x+3y+4z=5\,;$$

and (i) the origin, (ii) the point $(1, 2, 3)$.

Ans. (i) $5(x^2+y^2+z^2)-18x-27y-36z=0$;
(ii) $3(x^2+y^2+z^2)-2x-3y-4z-22=0$.

Ex. 7. The plane ABC, whose equation is $x/a+y/b+z/c=1$, meets the axes in A, B, C. Find equations to determine the circumcircle of the triangle ABC, and obtain the coordinates of its centre.

Ans. $x/a+y/b+z/c=1, \quad x^2+y^2+z^2-ax-by-cz=0$;

$$\frac{a(b^{-2}+c^{-2})}{2(a^{-2}+b^{-2}+c^{-2})},\ \frac{b(c^{-2}+a^{-2})}{2(a^{-2}+b^{-2}+c^{-2})},\ \frac{c(a^{-2}+b^{-2})}{2(a^{-2}+b^{-2}+c^{-2})}.$$

***Ex. 8.** If the axes are oblique, find the equation to the sphere whose centre is (x_1, y_1, z_1), and radius r.

Ans. $\Sigma(x-x_1)^2+2\Sigma(y-y_1)(z-z_1)\cos\lambda=r^2$.

***Ex. 9.** Prove that the necessary and sufficient conditions that the equation

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0,$$

referred to oblique axes, should represent a sphere, are

$$a=b=c=\frac{f}{\cos\lambda}=\frac{g}{\cos\mu}=\frac{h}{\cos\nu}.$$

Prove that the radius is $\dfrac{1}{a^2}\left(\dfrac{-\mathsf{S}}{\Delta}\right)^{\frac{1}{2}}$, where

$$\mathsf{S}\equiv\begin{vmatrix} a, & a\cos\nu, & a\cos\mu, & u \\ a\cos\nu, & a, & a\cos\lambda, & v \\ a\cos\mu, & a\cos\lambda, & a, & w \\ u, & v, & w, & d \end{vmatrix} \text{ and } \Delta\equiv\begin{vmatrix} 1, & \cos\nu, & \cos\mu \\ \cos\nu, & 1, & \cos\lambda \\ \cos\mu, & \cos\lambda, & 1 \end{vmatrix}.$$

57. Tangents and tangent planes. If P, (x_1, y_1, z_1) and Q, (x_2, y_2, z_2) are points on the sphere $x^2+y^2+z^2=a^2$, then

$$x_1^2+y_1^2+z_1^2=a^2=x_2^2+y_2^2+z_2^2,$$

and therefore

$$(x_1-x_2)(x_1+x_2)+(y_1-y_2)(y_1+y_2)+(z_1-z_2)(z_1+z_2)=0.$$

Now the direction-cosines of PQ are proportional to x_1-x_2, y_1-y_2, z_1-z_2; and if M is the mid-point of PQ and O is the origin, the direction-cosines of OM are proportional to x_1+x_2, y_1+y_2, z_1+z_2. Therefore PQ is at right angles

to OM. Suppose that OM meets the sphere in A and that PQ moves parallel to itself with its mid-point, M, on OA. Then when M is at A, PQ is a tangent to the sphere at A, and hence a tangent at A is at right angles to OA, and the locus of the tangents at A is the plane through A at right angles to OA. This plane is the **tangent plane** at A. The equation to the tangent plane at A, (α, β, γ), is

$$(x-\alpha)\alpha+(y-\beta)\beta+(z-\gamma)\gamma=0,$$

or

$$x\alpha+y\beta+z\gamma=\alpha^2+\beta^2+\gamma^2=a^2.$$

Ex. 1. Find the equation to the tangent plane at

$$(a\cos\theta\sin\phi,\quad a\sin\theta\sin\phi,\quad a\cos\phi)$$

to the sphere $x^2+y^2+z^2=a^2$.

Ans. $x\cos\theta\sin\phi+y\sin\theta\sin\phi+z\cos\phi=a.$

Ex. 2. Find the equation to the tangent plane at (x', y', z') to the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$.

Ans. $xx'+yy'+zz'+u(x+x')+v(y+y')+w(z+z')+d=0.$

Ex. 3. Find the condition that the plane $lx+my+nz=p$ should touch the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$.

Ans. $(ul+vm+wn+p)^2=(l^2+m^2+n^2)(u^2+v^2+w^2-d).$

Ex. 4. Find the equations to the spheres which pass through the circle $x^2+y^2+z^2=5$, $x+2y+3z=3$, and touch the plane $4x+3y=15$.

Ans. $x^2+y^2+z^2+2x+4y+6z-11=0,$

$5x^2+5y^2+5z^2-4x-8y-12z-13=0.$

Ex. 5. Prove that the tangent planes to the spheres

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0,$$
$$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$$

at any common point are at right angles if

$$2uu_1+2vv_1+2ww_1=d+d_1.$$

*58. **Radical plane of two spheres.** *If any secant through a given point* O *meets a given sphere in* P *and* Q, OP . OQ *is constant.*

The equations to the line through O, (α, β, γ), whose direction-cosines are l, m, n, are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}\quad(=r).$$

The point on this line, whose distance from O is r, has coordinates $\alpha+lr$, $\beta+mr$, $\gamma+nr$, and lies on the sphere

$$F(xyz) \equiv a(x^2+y^2+z^2)+2ux+2vy+2wz+d=0$$

if
$$ar^2+r\left(l\frac{\partial F}{\partial \alpha}+m\frac{\partial F}{\partial \beta}+n\frac{\partial F}{\partial \gamma}\right)+F(\alpha, \beta, \gamma)=0.$$

This equation gives the lengths of OP and OQ, and hence OP . OQ is given by $F(\alpha, \beta, \gamma)/a$, which is the same for all secants through O.

Definition. The measure of OP . OQ is **the power** of O with respect to the sphere.

If
$$S_1 \equiv x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0,$$
$$S_2 \equiv x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$$

are the equations to two spheres, the locus of points whose powers with respect to the spheres are equal is the plane given by

$$S_1=S_2, \text{ or } 2(u_1-u_2)x+2(v_1-v_2)y+2(w_1-w_2)z+d_1-d_2=0.$$

This plane is called the **radical plane** of the two spheres. It is evidently at right angles to the line joining the centres.

The radical planes of three spheres taken two by two pass through one line.

(The equations to the line are $S_1=S_2=S_3$.)

The radical planes of four spheres taken two by two pass through one point.

(The point is given by $S_1=S_2=S_3=S_4$.)

The equations to any two spheres can be put in the form

$$x^2+y^2+z^2+2\lambda_1x+d=0, \quad x^2+y^2+z^2+2\lambda_2x+d=0.$$

(Take the line joining the centres as x-axis and the radical plane as $x=0$.)

The equation $x^2+y^2+z^2+2\lambda x+d=0$, where λ is a parameter, represents a system of spheres any two of which have the same radical plane. The spheres are said to be *coaxal.*

Ex. 1. Prove that the members of the coaxal system intersect one another, touch one another, or do not intersect one another, according as $d \lesseqgtr 0$.

Ex. 2. Shew that the centres of the two spheres of the system which have zero-radius are at the points $(\pm\sqrt{d}, 0, 0)$. (These are the *limiting-points* of the system.)

Ex. 3. Shew that the equation $x^2+y^2+z^2+2\mu y+2\nu z-d=0$, where μ and ν are parameters, represents a system of spheres passing through the limiting points of the system $x^2+y^2+z^2+2\lambda x+d=0$, and cutting every member of that system at right angles.

Ex. 4. The locus of points whose powers with respect to two given spheres are in a constant ratio is a sphere coaxal with the two given spheres.

Ex. 5. Shew that the spheres which cut two given spheres along great circles all pass through two fixed points.

*Examples II.

1. A sphere of constant radius r passes through the origin, O, and cuts the axes (rectangular) in A, B, C. Prove that the locus of the foot of the perpendicular from O to the plane ABC is given by

$$(x^2+y^2+z^2)^2(x^{-2}+y^{-2}+z^{-2})=4r^2.$$

2. P is a variable point on a given line and A, B, C are its projections on the axes. Shew that the sphere OABC passes through a fixed circle.

3. A plane passes through a fixed point (a, b, c) and cuts the axes in A, B, C. Shew that the locus of the centre of the sphere OABC is

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2.$$

4. If the three diagonals of an octahedron intersect at right angles, the feet of the perpendiculars from the point of intersection to the faces of the octahedron lie on a sphere. If a, α; b, β; c, γ are the measures of the segments of the diagonals, the centre (ξ, η, ζ) of the sphere is given by

$$\frac{2\xi}{a^{-1}+\alpha^{-1}}=\frac{2\eta}{b^{-1}+\beta^{-1}}=\frac{2\zeta}{c^{-1}+\gamma^{-1}}=\frac{1}{(a\alpha)^{-1}+(b\beta)^{-1}+(c\gamma)^{-1}},$$

the diagonals being taken as coordinate axes. Prove that the points where the perpendiculars meet the opposite faces also lie on the sphere.

5. Prove that the locus of the centres of spheres which pass through a given point and touch a given plane is a conicoid.

6. Find the locus of the centres of spheres that pass through a given point and intercept a fixed length on a given straight line.

7. Find the locus of the centres of spheres of constant radius which pass through a given point and touch a given line.

8. Prove that the centres of spheres which touch the lines $y=mx$, $z=c$; $y=-mx$, $z=-c$, lie upon the conicoid $mxy+cz(1+m^2)=0$.

9. If the opposite edges of a tetrahedron are at right angles the centre of gravity is the mid-point of the line joining the point of concurrence of the perpendiculars and the centre of the circumscribing sphere.

10. If the opposite edges of a tetrahedron are at right angles the mid-points of the edges and the feet of the perpendiculars lie upon a sphere whose centre is the centre of gravity of the tetrahedron.

11. The sum of the squares of the intercepts made by a given sphere on any three mutually perpendicular lines through a fixed point is constant.

12. With any point P of a given plane as centre a sphere is described whose radius is equal to the tangent from P to a given sphere. Prove that all such spheres pass through two fixed points.

13. If $\lambda=\mu=\nu=\pi/3$, the plane and surface given by

$$x+y+z=0, \quad yz+zx+xy+a^2=0,$$

intersect in a circle of radius a.

14. If r is the radius of the circle

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0, \quad lx+my+nz=0,$$

prove that

$$(r^2+d)(l^2+m^2+n^2)=(mw-nv)^2+(nu-lw)^2+(lv-mu)^2.$$

15. Prove that the equations to the spheres that pass through the points (4, 1, 0), (2, −3, 4), (1, 0, 0), and touch the plane $2x+2y-z=11$, are

$$x^2+y^2+z^2-6x+2y-4z+5=0,$$

$$16x^2+16y^2+16z^2-102x+50y-49z+86=0.$$

16. Prove that the equation to a sphere, which lies in the octant OXYZ and touches the coordinate planes, is of the form

$$x^2+y^2+z^2-2\lambda(x+y+z)+2\lambda^2=0.$$

Prove that in general two spheres can be drawn through a given point to touch the coordinate planes, and find for what positions of the point the spheres are (i) real; (ii) coincident.

17. A is a point on OX and B on OY so that the angle OAB is constant ($=\alpha$). On AB as diameter a circle is described whose plane is parallel to OZ. Prove that as AB varies the circle generates the cone

$$2xy-z^2\sin 2\alpha=0.$$

18. POP′ is a variable diameter of the ellipse $z=0$, $x^2/a^2+y^2/b^2=1$, and a circle is described in the plane PP′ZZ′ on PP′ as diameter. Prove that as PP′ varies, the circle generates the surface

$$(x^2+y^2+z^2)(x^2/a^2+y^2/b^2)=x^2+y^2.$$

19. Prove that the equation to the sphere circumscribing the tetrahedron whose sides are

$$\frac{y}{b}+\frac{z}{c}=0, \quad \frac{z}{c}+\frac{x}{a}=0, \quad \frac{x}{a}+\frac{y}{b}=0, \quad \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,$$

is

$$\frac{x^2+y^2+z^2}{a^2+b^2+c^2}-\frac{x}{a}-\frac{y}{b}-\frac{z}{c}=0.$$

20. A variable plane is parallel to the given plane $x/a+y/b+z/c=0$, and meets the axes in A, B, C. Prove that the circle ABC lies on the cone

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0.$$

21. Find the locus of the centre of a variable sphere which passes through the origin O and meets the axes in A, B, C, so that the volume of the tetrahedron OABC is constant.

22. A sphere of constant radius k passes through the origin and meets the axes in A, B, C. Prove that the centroid of the triangle ABC lies on the sphere $9(x^2+y^2+z^2)=4k^2$.

23. The tangents drawn from a point P to a sphere are all equal to the distance of P from a fixed tangent plane to the sphere. Prove that the locus of P is a paraboloid of revolution.

24. Prove that the circles

$$x^2+y^2+z^2-2x+3y+4z-5=0, \quad 5y+6z+1=0;$$
$$x^2+y^2+z^2-3x-4y+5z-6=0, \quad x+2y-7z=0;$$

lie on the same sphere, and find its equation.

25. Find the conditions that the circles

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0, \quad lx+my+nz=p;$$
$$x^2+y^2+z^2+2u'x+2v'y+2w'z+d'=0, \quad l'x+m'y+n'z=p';$$

should lie on the same sphere.

26. OA, OB, OC are mutually perpendicular lines through the origin, and their direction-cosines are l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3. If OA$=a$, OB$=b$, OC$=c$, prove that the equation to the sphere OABC is

$$x^2+y^2+z^2-x(al_1+bl_2+cl_3)-y(am_1+bm_2+cm_3)-z(an_1+bn_2+cn_3)=0.$$

CHAPTER VI.

THE CONE.

59. Equation to a cone. A cone is a surface generated by a straight line which passes through a fixed point and intersects a given curve. If the given point O, say, be chosen as origin, the equation to the cone is homogeneous. For if P, (x', y', z') is any point on the cone, x', y', z' satisfy the equation. And since any point on OP is on the cone, and has coordinates (kx', ky', kz'), the equation is also satisfied by kx', ky', kz' for all values of k, and therefore must be homogeneous.

Cor. If $x/l = y/m = z/n$ is a generator of the cone represented by the homogeneous equation $f(x, y, z) = 0$, then $f(l, m, n) = 0$. Conversely, if the direction-ratios of a straight line which always passes through a fixed point satisfy a homogeneous equation, the line is a generator of a cone whose vertex is at the point.

Ex. 1. The line $x/l = y/m = z/n$, where $2l^2 + 3m^2 - 5n^2 = 0$, is a generator of the cone $2x^2 + 3y^2 - 5z^2 = 0$.

Ex. 2. Lines drawn through the point (α, β, γ) whose direction-ratios satisfy $al^2 + bm^2 + cn^2 = 0$ generate the cone

$$a(x - \alpha)^2 + b(y - \beta)^2 + c(z - \gamma)^2 = 0.$$

Ex. 3. Shew that the equation to the right circular cone whose vertex is O, axis OZ, and semi-vertical angle α, is $x^2 + y^2 = z^2 \tan^2\alpha$.

Ex. 4. The general equation to the cone of the second degree which passes through the axes is $fyz + gzx + hxy = 0$.

The general equation to the cone of the second degree is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

and this is to be satisfied by the direction-ratios of the axes, *i.e.* by 1, 0, 0; 0, 1, 0; 0, 0, 1.

Ex. 5. A cone of the second degree can be found to pass through any five concurrent lines.

Ex. 6. A cone of the second degree can be found to pass through any two sets of rectangular axes through the same origin.

Take one set as coordinate axes, and let the direction-cosines of the others be l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3. The equation to a cone containing the coordinate axes is $fyz+gzx+hxy=0$. If this cone also contains the first two axes of the second set,

$$fm_1n_1+gn_1l_1+hl_1m_1=0,$$

$$fm_2n_2+gn_2l_2+hl_2m_2=0.$$

Therefore, since $m_1n_1+m_2n_2+m_3n_3=0$, etc.,

$$fm_3n_3+gn_3l_3+hl_3m_3=0;$$

so that the cone contains the remaining axis.

Ex. 7. The equation to the cone whose vertex is the origin and which passes through the curve of intersection of the plane $lx+my+nz=p$ and the surface $ax^2+by^2+cz^2=1$ is

$$ax^2+by^2+cz^2=\left(\frac{lx+my+nz}{p}\right)^2.$$

Ex. 8. Find the equations to the cones with vertex at the origin which pass through the curves given by

(i) $x^2+y^2+z^2+2ax+b=0, \quad lx+my+nz=p$;

(ii) $ax^2+by^2=2z, \quad lx+my+nz=p$;

(iii) $x^2/a^2+y^2/b^2+z^2/c^2=1, \quad x^2/\alpha^2+y^2/\beta^2=2z$.

Ans. (i) $(x^2+y^2+z^2)p^2+2apx(lx+my+nz)+b(lx+my+nz)^2=0$;

(ii) $(ax^2+by^2)p=2z(lx+my+nz)$;

(iii) $4z^2(x^2/a^2+y^2/b^2+z^2/c^2)=(x^2/\alpha^2+y^2/\beta^2)^2$.

Ex. 9. The plane $x/a+y/b+z/c=1$ meets the coordinate axes in **A, B, C**. Prove that the equation to the cone generated by lines drawn from **O** to meet the circle **ABC** is

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0.$$

Ex. 10. Find the equation to the cone whose vertex is the origin and base the circle, $x=a$, $y^2+z^2=b^2$, and shew that the section of the cone by a plane parallel to the plane **XOY** is a hyperbola.

Ans. $a^2(y^2+z^2)=b^2x^2$.

Ex. 11. Shew that the equation to the cone whose vertex is the origin and base the curve $z=k$, $f(x, y)=0$ is $f\left(\frac{xk}{z}, \frac{yk}{z}\right)=0$.

60. Angle between lines in which a plane cuts a cone. We find it convenient to introduce here the following notation, to which we shall adhere throughout the book.

$$\mathsf{D} \equiv \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}.$$

$$\mathsf{A} \equiv \frac{\partial \mathsf{D}}{\partial a} = bc - f^2, \quad \mathsf{B} \equiv \frac{\partial \mathsf{D}}{\partial b} = ca - g^2, \quad \mathsf{C} \equiv \frac{\partial \mathsf{D}}{\partial c} = ab - h^2;$$

$$\mathsf{F} \equiv \frac{1}{2}\frac{\partial \mathsf{D}}{\partial f} = gh - af, \quad \mathsf{G} \equiv \frac{1}{2}\frac{\partial \mathsf{D}}{\partial g} = hf - bg, \quad \mathsf{H} \equiv \frac{1}{2}\frac{\partial \mathsf{D}}{\partial h} = fg - ch.$$

The student can easily verify that

$$\mathsf{BC} - \mathsf{F}^2 = a\mathsf{D}, \quad \mathsf{CA} - \mathsf{G}^2 = b\mathsf{D}, \quad \mathsf{AB} - \mathsf{H}^2 = c\mathsf{D};$$

$$\mathsf{GH} - \mathsf{AF} = f\mathsf{D}, \quad \mathsf{HF} - \mathsf{BG} = g\mathsf{D}, \quad \mathsf{FG} - \mathsf{CH} = h\mathsf{D}.$$

In what follows we use P^2 to denote

$$\begin{vmatrix} a, & h, & g, & u \\ h, & b, & f, & v \\ g, & f, & c, & w \\ u, & v, & w, & 0 \end{vmatrix},$$

or $$-(\mathsf{A}u^2 + \mathsf{B}v^2 + \mathsf{C}w^2 + 2\mathsf{F}vw + 2\mathsf{G}wu + 2\mathsf{H}uv).$$

The axes being rectangular to find the angle between the lines in which the plane $ux + vy + wz = 0$ *cuts the cone*

$$f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

If the line $x/l = y/m = z/n$ lies in the plane,

$$ul + vm + wn = 0; \quad \text{.....................(1)}$$

if it lies on the cone,

$$f(l, m, n) = 0. \quad \text{........................(2)}$$

Eliminate n between (1) and (2), and we obtain

$$l^2(cu^2 + aw^2 - 2gwu) + 2lm(hw^2 + cuv - fuw - gvw)$$
$$+ m^2(cv^2 + bw^2 - 2fvw) = 0. \quad \text{...........(3)}$$

Now the direction-cosines of the two lines of section satisfy the equations (1) and (2), and therefore they satisfy

equation (3). Therefore if they are l_1, m_1, n_1; l_2, m_2, n_2;

$$\frac{l_1 l_2}{bw^2+cv^2-2fvw}=\frac{m_1 m_2}{cu^2+aw^2-2gwu}$$

$$=\frac{l_1 m_2+l_2 m_1}{-2(hw^2+cuv-fuw-gvw)}$$

$$=\frac{l_1 m_2-l_2 m_1}{\pm 2\{(hw^2+cuv\ldots)^2-(bw^2\ldots)(cu^2\ldots)\}^{\frac{1}{2}}} \quad\ldots\ldots\ldots\ldots(4)$$

$$=\frac{l_1 m_2-l_2 m_1}{\pm 2w\mathsf{P}}.$$

From the symmetry, each of the expressions in (4) is seen to be equal to

$$\frac{n_1 n_2}{av^2+bu^2-2huv}=\frac{m_1 n_2-m_2 n_1}{\pm 2u\mathsf{P}}=\frac{n_1 l_2-n_2 l_1}{\pm 2v\mathsf{P}}$$

But if θ is the angle between the lines,

$$\frac{\cos\theta}{l_1 l_2+m_1 m_2+n_1 n_2}=\frac{\sin\theta}{\{\Sigma(m_1 n_2-m_2 n_1)^2\}^{\frac{1}{2}}}.$$

$$\therefore \frac{\cos\theta}{(a+b+c)(u^2+v^2+w^2)-f(u,v,w)}=\frac{\sin\theta}{\pm 2(u^2+v^2+w^2)^{\frac{1}{2}}\mathsf{P}}.$$

Ex. 1. Find the equations to the lines in which the plane $2x+y-z=0$ cuts the cone $4x^2-y^2+3z^2=0$.

Ans. $\frac{x}{-1}=\frac{y}{4}=\frac{z}{2}$; $\frac{x}{1}=\frac{y}{-2}=\frac{z}{0}$.

Ex. 2. Find the angles between the lines of section of the following planes and cones:

(i) $6x-10y-7z=0$, $108x^2-20y^2-7z^2=0$;

(ii) $3x+y+5z=0$, $6yz-2zx+5xy=0$;

(iii) $2x-3y+z=0$, $3x^2-5y^2-7z^2+36yz-20zx-2xy=0$.

Ans. (i) $\cos^{-1}\frac{16}{21}$, (ii) $\cos^{-1}\frac{1}{6}$, (iii) $\cos^{-1}\frac{5}{\sqrt{39}}$.

Ex. 3. Prove that the plane $ax+by+cz=0$ cuts the cone $yz+zx+xy=0$ in perpendicular lines if

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0.$$

61. Condition of tangency of plane and cone. If $P=0$,

or $$Au^2+Bv^2+Cw^2+2Fvw+2Gwu+2Huv=0, \quad\ldots\ldots(1)$$

then $\sin\theta=0$, and therefore the lines of section coincide, or the plane touches the cone. Equation (1) shews that the line $\dfrac{x}{u}=\dfrac{y}{v}=\dfrac{z}{w}$, *i.e.* the normal through O to the plane, is a generator of the cone

$$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0. \quad\ldots\ldots\ldots(2)$$

Similarly, since we have $BC-F^2=aD$, and the corresponding equations at the head of paragraph 60, it follows that a normal through the origin to a tangent plane to the cone (2) is a generator of the cone

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0,$$

i.e. of the given cone. The two cones are therefore such that each is the locus of the normals drawn through the origin to the tangent planes to the other, and they are on that account said to be *reciprocal.*

Ex. 1. Prove that the cones $ax^2+by^2+cz^2=0$ and $\dfrac{x^2}{a}+\dfrac{y^2}{b}+\dfrac{z^2}{c}=0$ are reciprocal.

Ex. 2. Prove that tangent planes to the cone $lyz+mzx+nxy=0$ are at right angles to generators of the cone

$$l^2x^2+m^2y^2+n^2z^2-2mnyz-2nlzx-2lmxy=0.$$

Ex. 3. Prove that perpendiculars drawn from the origin to tangent planes to the cone

$$3x^2+4y^2+5z^2+2yz+4zx+6xy=0$$

lie on the cone $19x^2+11y^2+3z^2+6yz-10zx-26xy=0.$

Ex. 4. Shew that the general equation to a cone which touches the coordinate planes is $a^2x^2+b^2y^2+c^2z^2-2bcyz-2cazx-2abxy=0.$

62. Condition that the cone has three mutually perpendicular generators. The condition that the plane should cut the cone in perpendicular generators is

$$(a+b+c)(u^2+v^2+w^2)=f(u, v, w). \quad\ldots\ldots\ldots\ldots(1)$$

If also the normal to the plane lies on the cone, we have

$$f(u, v, w)=0,$$

and therefore $$a+b+c=0.$$

In this case the cone has three mutually perpendicular generators, viz., the normal to the plane and the two perpendicular lines in which the plane cuts the cone.

If $a+b+c=0$, the cone has an infinite number of sets of mutually perpendicular generators. For if $ux+vy+wz=0$ be any plane whose normal lies on the cone, then

$$f(u, v, w)=0,$$

and therefore $(a+b+c)(u^2+v^2+w^2)=f(u, v, w)$,

since $$a+b+c=0.$$

Hence, by (1), the plane cuts the cone in perpendicular generators. Thus any plane through the origin which is normal to a generator of the cone cuts the cone in perpendicular lines, or there are two generators of the cone at right angles to one another, and at right angles to any given generator.

Ex. 1. If a right circular cone has three mutually perpendicular generators, the semi-vertical angle is $\tan^{-1}\sqrt{2}$. (Cf. Ex. 3, § 59.)

Ex. 2. Shew that the cone whose vertex is at the origin and which passes through the curve of intersection of the sphere $x^2+y^2+z^2=3a^2$, and any plane at a distance a from the origin, has three mutually perpendicular generators.

Ex. 3. Prove that the cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ has three mutually perpendicular tangent planes if

$$bc+ca+ab=f^2+g^2+h^2.$$

Ex. 4. If $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ represent one of a set of three mutually perpendicular generators of the cone $5yz-8zx-3xy=0$, find the equations to the other two.

Ans. $x=y=-z$, $4x=-5y=20z$.

Ex. 5. Prove that the plane $lx+my+nz=0$ cuts the cone

$$(b-c)x^2+(c-a)y^2+(a-b)z^2+2fyz+2gzx+2hxy=0$$

in perpendicular lines if

$$(b-c)l^2+(c-a)m^2+(a-b)n^2+2fmn+2gnl+2hlm=0.$$

63. Equation to cone with given conic for base. *To find the equation to the cone whose vertex is the point (α, β, γ) and base the conic*

$$f(x, y)\equiv ax^2+2hxy+by^2+2gx+2fy+c=0, \quad z=0.$$

The equations to any line through (α, β, γ) are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=z-\gamma,$$

and the line meets the plane $z=0$ in the point

$$(\alpha-l\gamma,\ \beta-m\gamma,\ 0).$$

This point is on the given conic if $f(\alpha-l\gamma, \beta-m\gamma)=0$,

i.e. if $$f(\alpha, \beta)-\gamma\left(l\frac{\partial f}{\partial \alpha}+m\frac{\partial f}{\partial \beta}\right)+\gamma^2\phi(l, m)=0, \quad \text{.........(1)}$$

where $\phi(x, y)\equiv ax^2+2hxy+by^2$. If we eliminate l and m between the equations to the line and (1), we obtain the equation to the locus of lines which pass through (α, β, γ) and intersect the conic, *i.e.* the equation to the cone. The result is

$$f(\alpha, \beta)-\gamma\left(\frac{x-\alpha}{z-\gamma}\frac{\partial f}{\partial \alpha}+\frac{y-\beta}{z-\gamma}\frac{\partial f}{\partial \beta}\right)+\gamma^2\phi\left(\frac{x-\alpha}{z-\gamma}, \frac{y-\beta}{z-\gamma}\right)=0,$$

i.e. $$(z-\gamma)^2 f(\alpha, \beta)-\gamma(z-\gamma)\left(\overline{x-\alpha}\frac{\partial f}{\partial \alpha}+\overline{y-\beta}\frac{\partial f}{\partial \beta}\right)$$
$$+\gamma^2\phi(x-\alpha, y-\beta)=0.$$

This equation may be transformed as follows:

The coefficient of γ^2 is

$$f(\alpha, \beta)+(x-\alpha)\frac{\partial f}{\partial \alpha}+(y-\beta)\frac{\partial f}{\partial \beta}+\phi(x-\alpha, y-\beta)$$
$$=f(\alpha+\overline{x-\alpha},\ \beta+\overline{y-\beta})=f(x, y);$$

and the coefficient of $-z\gamma$ is

$$(x-\alpha)\frac{\partial f}{\partial \alpha}+(y-\beta)\frac{\partial f}{\partial \beta}+2f(\alpha, \beta).$$

If $f(x, y)$ be made homogeneous by means of an auxiliary variable t which is equated to unity after differentiation, we have, by Euler's theorem,

$$\alpha\frac{\partial f}{\partial \alpha}+\beta\frac{\partial f}{\partial \beta}+t\frac{\partial f}{\partial t}=2f(\alpha, \beta, t).$$

Therefore the coefficient of $-z\gamma$ becomes

$$x\frac{\partial f}{\partial \alpha}+y\frac{\partial f}{\partial \beta}+t\frac{\partial f}{\partial t}.$$

Hence the equation to the cone is

$$z^2 f(\alpha, \beta) - z\gamma\left(x\frac{\partial f}{\partial \alpha} + y\frac{\partial f}{\partial \beta} + t\frac{\partial f}{\partial t}\right) + \gamma^2 f(x, y) = 0.$$

It is to be noted that by equating to zero the coefficient of $z\gamma$, we obtain the equation to the polar of $(\alpha, \beta, 0)$ with respect to the given conic.

(The above method is given by de Longchamps, *Problèmes de Géométrie Analytique*, vol. iii.)

Ex. 1. Find the equation to the cone whose vertex is (α, β, γ) and base (i) $ax^2+by^2=1$, $z=0$; (ii) $y^2=4ax$, $z=0$.

Ans. (i) $z^2(a\alpha^2+b\beta^2-1)-2z\gamma(a\alpha x+b\beta y-1)+\gamma^2(ax^2+by^2-1)=0$;

(ii) $z^2(\beta^2-4a\alpha)-2z\gamma\{\beta y-2a(x+\alpha)\}+\gamma^2(y^2-4ax)=0$.

Ex. 2. Find the locus of points from which three mutually perpendicular lines can be drawn to intersect the conic $z=0$, $ax^2+by^2=1$.

(If (α, β, γ) is on the locus, the cone, Ex. 1 (i), has three mutually perpendicular generators.)

Ans. $ax^2+by^2+z^2(a+b)=1$.

Ex. 3. Shew that the locus of points from which three mutually perpendicular lines can be drawn to intersect a given circle is a surface of revolution.

Ex. 4. A cone has as base the circle $z=0$, $x^2+y^2+2ax+2by=0$, and passes through the fixed point $(0, 0, c)$ on the z-axis. If the section of the cone by the plane **ZOX** is a rectangular hyperbola, prove that the vertex lies on a fixed circle.

Ex. 5. Prove that the locus of points from which three mutually perpendicular planes can be drawn to touch the ellipse $x^2/a^2+y^2/b^2=1$, $z=0$, is the sphere $x^2+y^2+z^2=a^2+b^2$.

*Examples III.

1. Shew that the bisectors of the angles between the lines in which the plane $ux+vy+wz=0$ cuts the cone $ax^2+by^2+cz^2=0$ lie on the cone

$$\frac{u(b-c)}{x}+\frac{v(c-a)}{y}+\frac{w(a-b)}{z}=0.$$

[Five concurrent lines are necessary to determine a cone of the second degree, and the form of the given result shews that the required cone is to pass through the coordinate axes and the two bisectors. Assume, therefore, that the required equation is

$$fyz+gzx+hxy=0. \quad \text{.............................(1)}$$

The given cone is $\qquad ax^2+by^2+cz^2=0.$ ....(2)

The necessary and sufficient conditions that the cone (1) should contain the bisectors may be stated, (i) the plane $ux+vy+wz=0$

must cut the cone (1) in perpendicular lines; (ii) the lines of section of the plane and the cones (1) and (2) must be harmonically conjugate. From (i),

$$fvw+gwu+huv=0. \quad\text{..........................}(3)$$

Again, four lines are harmonically conjugate if their projections on any plane are harmonically conjugate, and the equations to the projections on $z=0$ are obtained by eliminating z between the equations to the plane and cones, and hence they are

$$x^2(aw^2+cu^2)+2cuvxy+y^2(bw^2+cv^2)=0, \quad z=0;$$

$$gux^2+xy(fu+gv-wh)+fvy^2=0, \quad z=0.$$

Therefore the condition (ii) gives

$$fv(aw^2+cu^2)+gu(bw^2+cv^2)=cuv(fu+gv-wh);$$

(cf. Smith, *Conic Sections*, p. 55.)

$$i.e. \quad afvw+bgwu+chuv=0. \quad\text{.......................}(4)$$

From (3) and (4), we obtain

$$\frac{f}{u(b-c)}=\frac{g}{v(c-a)}=\frac{h}{w(a-b)}.]$$

2. Shew that the bisectors in Ex. 1 also lie on the cone

$$\Sigma u^2x^2\{-(b-c)u^2+(c-a)v^2+(a-b)w^2\}=0.$$

3. Two cones pass through the curves $y=0$, $z^2=4ax$; $x=0$, $z^2=4by$, and they have a common vertex; the plane $z=0$ meets them in two conics that intersect in four concyclic points. Shew that the vertex lies on the surface $z^2(x/a+y/b)=4(x^2+y^2)$.

4. Planes through OX and OY include an angle α. Shew that their line of intersection lies on the cone $z^2(x^2+y^2+z^2)=x^2y^2\tan^2\alpha$.

5. Any plane whose normal lies on the cone

$$(b+c)x^2+(c+a)y^2+(a+b)z^2=0$$

cuts the surface $ax^2+by^2+cz^2=1$ in a rectangular hyperbola

6. Find the angle between the lines given by

$$x+y+z=0, \quad \frac{yz}{b-c}+\frac{zx}{c-a}+\frac{xy}{a-b}=0.$$

7. Shew that the angle between the lines given by

$$x+y+z=0, \quad ayz+bzx+cxy=0$$

is $\pi/2$ if $a+b+c=0$, but $\pi/3$ if $1/a+1/b+1/c=0$.

8. Shew that the plane $ax+by+cz=0$ cuts the cone

$$yz+zx+xy=0$$

in two lines inclined at an angle

$$\tan^{-1}\left[\frac{\{(a^2+b^2+c^2)(a^2+b^2+c^2-2bc-2ca-2ab)\}^{\frac{1}{2}}}{bc+ca+ab}\right],$$

and by considering the value of this expression when $a+b+c=0$, shew that the cone is of revolution, and that its axis is $x=y=z$ and vertical angle $\tan^{-1}2\sqrt{2}$.

9. The axes being rectangular, prove that the cone

$$x^2=2(y^2+z^2)$$

contains an infinite number of sets of three generators mutually inclined at an angle $\pi/3$.

10. Through a fixed point O a line is drawn to meet three fixed intersecting planes in P, Q, R. If PQ : PR is constant, prove that the locus of the line is a cone whose vertex is O.

11. The vertex of a cone is (a, b, c) and the yz-plane cuts it in the curve $F(y, z)=0$, $x=0$. Shew that the zx-plane cuts it in the curve $y=0$, $F\left(\frac{bx}{x-a}, \frac{cx-az}{x-a}\right)=0$.

12. OP and OQ are two straight lines that remain at right angles and move so that the plane OPQ always passes through the z-axis. If OP describes the cone $F(y/x, z/x)=0$, prove that OQ describes the cone

$$F\left\{\frac{y}{x}, \left(-\frac{x}{z}-\frac{y^2}{zx}\right)\right\}=0.$$

13. Prove that $ax^2+by^2+cz^2+2ux+2vy+2wz+d=0$ represents a cone if $u^2/a+v^2/b+w^2/c=d$.

14. Prove that if

$$F(xyz)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$$

represents a cone, the coordinates of the vertex satisfy the equations $F_x=0$, $F_y=0$, $F_z=0$, $F_t=0$, where t is used to make $F(x, y, z)$ homogeneous and is equated to unity after differentiation.

15. Prove that the equations

$$2y^2-8yz-4zx-8xy+6x-4y-2z+5=0,$$

$$2x^2+2y^2+7z^2-10yz-10zx+2x+2y+26z-17=0,$$

represent cones whose vertices are $(-7/6, 1/3, 5/6)$, $(2, 2, 1)$.

16. Find the conditions that the lines of section of the plane $lx+my+nz=0$ and the cones $fyz+gzx+hxy=0$, $ax^2+by^2+cz^2=0$, should be coincident.

$$\left(\frac{bn^2+cm^2}{fmn}=\frac{cl^2+an^2}{gnl}=\frac{am^2+bl^2}{hlm}.\right)$$

17. Find the equations to the planes through the z-axis and the lines of section of the plane $ux+vy+wz=0$ and cone $f(x, y, z)=0$, and prove that the plane touches the cone if $P=0$. (The axes may be oblique.)

18. Prove that the equation to the cone through the coordinate axes and the lines of section of the cone $11x^2-5y^2+z^2=0$ and the plane $7x-5y+z=0$ is $14yz-30zx+3xy=0$, and that the other common generators of the two cones lie in the plane $11x+7y+7z=0$.

19. Prove that the common generators of the cones

$$(b^2c^2-a^4)x^2+(c^2a^2-b^4)y^2+(a^2b^2-c^4)z^2=0,$$

$$\frac{bc-a^2}{ax}+\frac{ca-b^2}{by}+\frac{ab-c^2}{cz}=0,$$

lie in the planes

$$(bc\pm a^2)x+(ca\pm b^2)y+(ab\pm c^2)z=0.$$

20. Prove that the equation to the cone through the coordinate axes and the lines in which the plane $lx+my+nz=0$ cuts the cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ is

$$l(bn^2+cm^2-2fmn)yz+m(cl^2+an^2-2gnl)zx+n(am^2+bl^2-2hlm)xy=0.$$

21. Prove that the equation $\sqrt{fx}+\sqrt{gy}+\sqrt{hz}=0$ represents a cone that touches the coordinate planes, and that the equation to the reciprocal cone is $fyz+gzx+hxy=0$.

22. Prove that the equation to the planes through the origin perpendicular to the lines of section of the plane $lx+my+nz=0$ and the cone $ax^2+by^2+cz^2=0$ is

$$x^2(bn^2+cm^2)+y^2(cl^2+an^2)+z^2(am^2+bl^2)-2amnyz-2bnlzx-2clmxy=0.$$

CHAPTER VII.

THE CENTRAL CONICOIDS.

64. The locus of the equation

$$(1)\qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

$$(2)\qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

$$(3)\qquad -\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

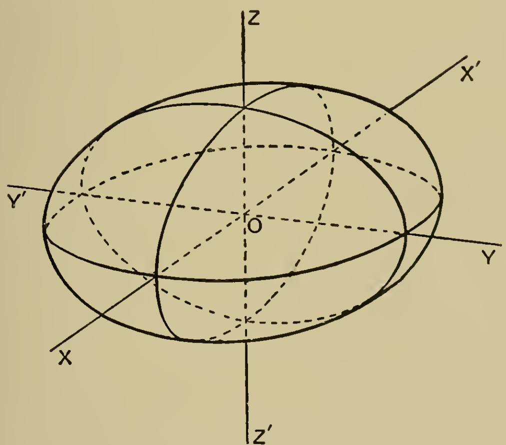

FIG. 29

We have shewn in § 9 that the equation (1) represents the surface generated by the variable ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{k^2}{c^2},\quad z=k,$$

whose centre moves along Z′OZ, and passes in turn through every point between $(0, 0, -c)$ and $(0, 0, +c)$. The surface is the **ellipsoid**, and is represented in fig. 29. The section by any plane parallel to a coordinate plane is an ellipse.

Similarly, we might shew that the surface represented by equation (2) is generated by a variable ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{k^2}{c^2}, \quad z=k,$$

whose centre moves on Z′OZ, passing in turn through every point on it. The surface is the **hyperboloid of one sheet**, and is represented in fig. 30. The section by any plane parallel to one of the coordinate planes YOZ or ZOX is a hyperbola.

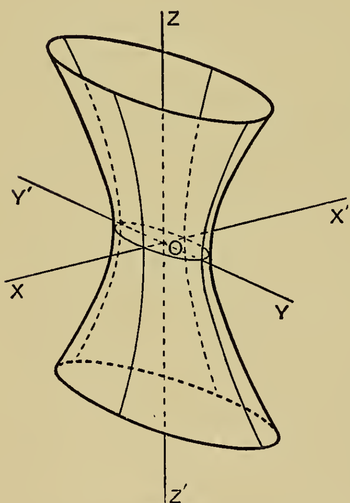

FIG. 30.

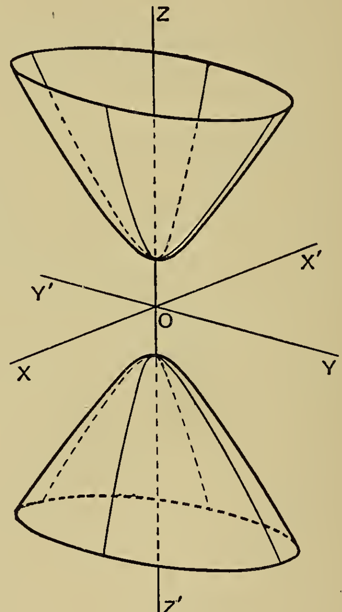

FIG. 31.

The surface given by equation (3) is also generated by a variable ellipse whose centre moves on Z′OZ. The ellipse is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{k^2}{c^2}-1, \quad z=k,$$

and is imaginary if $-c<k<c$; hence no part of the surface lies between the planes $z=\pm c$.

The surface is the **hyperboloid of two sheets**, and is represented in fig. 31. The section by any plane parallel to one of the coordinate planes YOZ, ZOX is a hyperbola.

If (x', y', z') is any point on one of these surfaces, $(-x', -y', -z')$ is also on it; hence the origin bisects all chords of the surface which pass through it. The origin is the only point which possesses this property, and is called **the centre**. The surfaces are called **the central conicoids.**

65. Diametral planes and conjugate diameters. An equation of the form

$$ax^2+by^2+cz^2=1$$

represents a central conicoid. The equations to any line parallel to OX are $y=\lambda$, $z=\mu$, and it meets the surface in the points

$$\left(\pm\sqrt{\frac{1-b\lambda^2-c\mu^2}{a}},\ \lambda,\ \mu\right),$$

and hence the plane YOZ bisects all chords parallel to OX. Any chord of the conicoid which passes through the centre is a **diameter**, and the plane which bisects a system of parallel chords is a **diametral plane**. Thus YOZ is the diametral plane which bisects chords parallel to OX, or shortly, is the diametral plane of OX. Similarly, the diametral planes ZOX, XOY bisect chords parallel to OY and OZ respectively. The three diametral planes YOZ, ZOX, XOY are such that each bisects chords parallel to the line of intersection of the other two. They are called **conjugate diametral planes**. The diameters X′OX, Y′OY, Z′OZ are such that the plane through any two bisects chords parallel to the third. They are called **conjugate diameters.**

If the axes are rectangular, the diametral planes YOZ, ZOX, XOY are at right angles to the chords which they bisect. Diametral planes which are at right angles to the chords which they bisect are **principal planes**. The lines of intersection of principal planes are **principal axes**. Hence if the axes are rectangular the equation $ax^2+by^2+cz^2=1$

represents a central conicoid referred to its principal axes as coordinate axes.

66. *A line through a given point* A, (α, β, γ) *meets a central conicoid* $ax^2+by^2+cz^2=1$ *in* P *and* Q; *to find the lengths of* AP *and* AQ.

If l, m, n are the direction-ratios of a line through A, the coordinates of the point on it whose distance from A is r are $\alpha+lr, \beta+mr, \gamma+nr$. If this point is on the conicoid,

$$r^2(al^2+bm^2+cn^2)+2r(a\alpha l+b\beta m+c\gamma n)$$
$$+a\alpha^2+b\beta^2+c\gamma^2-1=0. \quad\text{........(1)}$$

This equation gives two values of r which are the measures of AP and AQ.

Ex. 1. If OD is the diameter parallel to APQ, AP . AQ : OD^2 is constant.

Ex. 2. If DOD′ is any diameter of the conicoid and OR and OR′ are the diameters parallel to AD and AD′, $\dfrac{\text{AD}^2}{\text{OR}^2}+\dfrac{\text{AD}'^2}{\text{OR}'^2}$ is constant.

Ex. 3. If AD, AD′ meet the conicoid again in E and E′, $\dfrac{\text{AD}}{\text{AE}}+\dfrac{\text{AD}'}{\text{AE}'}$ is constant.

67. Tangents and tangent planes. If $a\alpha^2+b\beta^2+c\gamma^2=1$, the point A, (α, β, γ) is on the conicoid; one of the values of r given by the equation (1) of § 66 is zero, and A coincides with one of the points P or Q, say P.

If, also, $$a\alpha l+b\beta m+c\gamma n=0,$$
the two values of r given by the equation are zero, *i.e.* P and Q coincide at the point (α, β, γ) on the surface, and the line APQ is a **tangent** to the conicoid at A. Hence, if A, (α, β, γ) is a point on the surface, the condition that the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad\text{....................(2)}$$

should be a tangent at A, is

$$a\alpha l+b\beta m+c\gamma n=0. \quad\text{....................(3)}$$

If we eliminate l, m, n between (2) and (3), we obtain the equation to the locus of all the tangent lines through (α, β, γ), viz.,

$$(x-\alpha)a\alpha+(y-\beta)b\beta+(z-\gamma)c\gamma=0,$$
or
$$a\alpha x+b\beta y+c\gamma z=1.$$

Hence the tangent lines at (α, β, γ) lie in the plane

$$a\alpha x + b\beta y + c\gamma z = 1,$$

which is the **tangent plane at** (α, β, γ).

68. *To find the condition that the plane $lx + my + nz = p$ should touch the conicoid $ax^2 + by^2 + cz^2 = 1$.*

If the point of contact is (α, β, γ), the given plane is represented by the two equations

$$a\alpha x + b\beta y + c\gamma z = 1,$$
$$lx + my + nz = p.$$

Therefore $$\alpha = \frac{l}{ap}, \quad \beta = \frac{m}{bp}, \quad \gamma = \frac{n}{cp};$$

and, since (α, β, γ) is on the conicoid,

$$\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2.$$

Cor. The two tangent planes which are parallel to $lx + my + nz = 0$ are given by

$$lx + my + nz = \pm\sqrt{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}}.$$

Ex. 1. Find the locus of the point of intersection of three mutually perpendicular tangent planes to a central conicoid.

If the axes are rectangular and

$$l_r x + m_r y + n_r z = \sqrt{\frac{l_r^2}{a} + \frac{m_r^2}{b} + \frac{n_r^2}{c}}, \quad r = 1, 2, 3,$$

represent three mutually perpendicular tangent planes, squaring and adding, we obtain

$$x^2 + y^2 + z^2 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Hence the common point of the planes lies on a sphere concentric with the conicoid. (It is called *the director sphere.*)

Ex. 2. Prove that the equation to the two tangent planes to the conicoid $ax^2 + by^2 + cz^2 = 1$ which pass through the line

$$u \equiv lx + my + nz - p = 0, \quad u' \equiv l'x + m'y + n'z - p' = 0, \text{ is}$$

$$u^2\left(\frac{l'^2}{a} + \frac{m'^2}{b} + \frac{n'^2}{c} - p'^2\right) - 2uu'\left(\frac{ll'}{a} + \frac{mm'}{b} + \frac{nn'}{c} - pp'\right)$$
$$+ u'^2\left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} - p^2\right) = 0.$$

(Use the condition that $u + \lambda u' = 0$ should be a tangent plane.)

Ex. 3. Find the equations to the tangent planes to

$$2x^2-6y^2+3z^2=5$$

which pass through the line $x+9y-3z=0=3x-3y+6z-5$.

Ans. $2x-12y+9z=5$, $4x+6y+3z=5$.

Ex. 4. A pair of perpendicular tangent planes to the ellipsoid whose equation, referred to rectangular axes, is $x^2/a^2+y^2/b^2+z^2/c^2=1$, passes through the fixed point $(0, 0, k)$. Shew that their line of intersection lies on the cone

$$x^2(b^2+c^2-k^2)+y^2(c^2+a^2-k^2)+(z-k)^2(a^2+b^2)=0.$$

Ex. 5. Tangent planes are drawn to the conicoid $ax^2+by^2+cz^2=1$ through the point (α, β, γ). Prove that the perpendiculars to them from the origin generate the cone $(\alpha x+\beta y+\gamma z)^2=\dfrac{x^2}{a}+\dfrac{y^2}{b}+\dfrac{z^2}{c}$.

Prove that the reciprocal of this cone is the cone

$$(ax^2+by^2+cz^2)(a\alpha^2+b\beta^2+c\gamma^2-1)-(a\alpha x+b\beta y+c\gamma z)^2=0,$$

and hence shew that the tangent planes envelope the cone

$$(ax^2+by^2+cz^2-1)(a\alpha^2+b\beta^2+c\gamma^2-1)-(a\alpha x+b\beta y+c\gamma z-1)^2=0.$$

69. The polar plane. We now proceed to define the polar of a point with respect to a conicoid, and to find its equation.

Definition. If any secant, APQ, through a given point A, meets a conicoid in P and Q, then the locus of R, the harmonic conjugate of A with respect to P and Q, is the polar of A with respect to the conicoid.

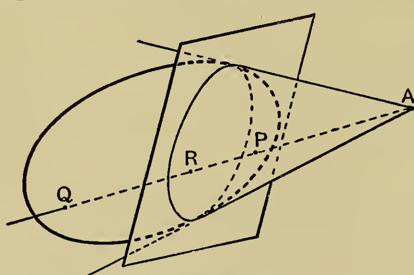

FIG. 32.

Let A, R (fig. 32) be the points (α, β, γ), (ξ, η, ζ), and let APQ have direction-ratios l, m, n. Then the equations to APQ are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n},$$

and, as in § 66, r_1, r_2, the measures of AP and AQ are the roots of

$$r^2(al^2+bm^2+cn^2)+2r(a\alpha l+b\beta m+c\gamma n)$$
$$+(a\alpha^2+b\beta^2+c\gamma^2-1)=0.$$

Let ρ be the measure of AR. Then, since AP, AR, AQ are in harmonic progression,

$$\rho=\frac{2r_1r_2}{r_1+r_2}=-\frac{a\alpha^2+b\beta^2+c\gamma^2-1}{a\alpha l+b\beta m+c\gamma n}.$$

And from the equations to the line

$$\xi-\alpha=l\rho,\quad \eta-\beta=m\rho,\quad \zeta-\gamma=n\rho,$$

therefore

$$(\xi-\alpha)a\alpha+(\eta-\beta)b\beta+(\zeta-\gamma)c\gamma=-(a\alpha^2+b\beta^2+c\gamma^2-1).$$

Hence the locus of (ξ, η, ζ) is the plane given by

$$a\alpha x+b\beta y+c\gamma z=1,$$

which is called the **polar plane** of (α, β, γ).

Cor. If A is on the surface, the polar plane of A is the tangent plane at A.

The student cannot have failed to notice the similarity between the equations to corresponding loci in the plane and in space. There is a close analogy between the equations to the line and the plane, the circle and the sphere, the ellipse and the ellipsoid, the tangent or polar and the tangent plane or polar plane. Examples of this analogy will constantly recur, and it is well to note these and make use of the analogy as an aid to remember useful results.

70. Polar lines. It is evident that if the polar plane of (α, β, γ) passes through (ξ, η, ζ), then the polar plane of (ξ, η, ζ) passes through (α, β, γ). Hence if the polar plane of any point on a line AB passes through a line PQ, then the polar plane of any point on PQ passes through that point on AB, and therefore passes through AB. The lines AB and PQ are then said to be **polar lines** with respect to the conicoid.

The polar plane of $(\alpha+lr, \beta+mr, \gamma+nr)$, any point on the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad\text{.....................(1)}$$

is $\quad a\alpha x+b\beta y+c\gamma z-1+r(alx+bmy+cnz)=0.$

and, evidently, for all values of r, passes through the line

$$a\alpha x+b\beta y+c\gamma z-1=0=alx+bmy+cnz.$$

This is therefore the polar of the line (1).

Ex. 1. If P, (x_1, y_1, z_1), Q, (x_2, y_2, z_2) are any points, the polar of PQ with respect to $ax^2+by^2+cz^2=1$ is given by

$$axx_1+byy_1+czz_1=1, \quad axx_2+byy_2+czz_2=1.$$

(Hence if P and Q are on the conicoid the polar of PQ is the line of intersection of the tangent planes at P and Q.)

Ex. 2. Prove that the polar of a given line is the chord of contact of the two tangent planes through the line.

Ex. 3. Find the equations to the polar of the line

$$-2x=25y-1=2z$$

with respect to the conicoid $2x^2-25y^2+2z^2=1$. Prove that it meets the conicoid in two real points P and Q, and verify that the tangent planes at P and Q pass through the given line.

Ans. $\dfrac{x}{1}=\dfrac{y+1}{0}=\dfrac{z+1}{1}$.

Ex. 4. Find the locus of straight lines drawn through a fixed point (α, β, γ) at right angles to their polars with respect to $ax^2+by^2+cz^2=1$; (rectangular axes).

Ans. $\Sigma\dfrac{\alpha}{x-\alpha}\left(\dfrac{1}{b}-\dfrac{1}{c}\right)=0.$

Ex. 5. Prove that lines through (α, β, γ) at right angles to their polars with respect to $\dfrac{x^2}{a+b}+\dfrac{y^2}{2a}+\dfrac{z^2}{2b}=1$ generate the cone

$$(y-\beta)(\alpha z-\gamma x)+(z-\gamma)(\alpha y-\beta x)=0.$$

What is the peculiarity of the case when $a=b$?

Ex. 6. Find the conditions that the lines

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}, \quad \frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'},$$

should be polar with respect to the conicoid $ax^2+by^2+cz^2=1$.

Ans. $\Sigma a\alpha\alpha'=1$, $\Sigma a\alpha' l=0$, $\Sigma a\alpha l'=0$, $\Sigma all'=0$.

Ex. 7. Find the condition that the line $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$ should intersect the polar of the line $\dfrac{x-\alpha'}{l'}=\dfrac{y-\beta'}{m'}=\dfrac{z-\gamma'}{n'}$ with respect to the conicoid $ax^2+by^2+cz^2=1$.

Ans. $(a\alpha l'+b\beta m'+c\gamma n')(a\alpha' l+b\beta' m+c\gamma' n)$
$$=(all'+bmm'+cnn')(a\alpha\alpha'+b\beta\beta'+c\gamma\gamma'-1).$$

Ex. 8. Prove that if AB intersects the polar of PQ, then PQ intersects the polar of AB. (AB and PQ are then said to be **conjugate** with respect to the conicoid.)

71. Section with a given centre. If (α, β, γ) is the mid-point of the chord whose equations are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}, \qquad \text{.....................(1)}$$

the equation (1) of § 66 is of the form $r^2=k^2$, and therefore

$$a\alpha l+b\beta m+c\gamma n=0. \qquad \text{.....................(2)}$$

Hence all chords which are bisected at (α, β, γ) lie in the plane

$$(x-\alpha)a\alpha+(y-\beta)b\beta+(z-\gamma)c\gamma=0.$$

This plane meets the surface in a conic of which (α, β, γ) is the centre.

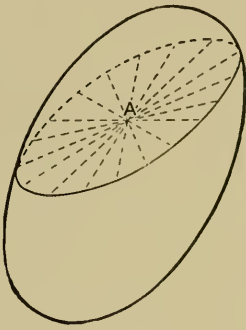

FIG. 33.

Compare the equation to the chord of the conic $ax^2+by^2=1$ whose mid-point is (α, β).

Ex. 1. Find the equation to the plane which cuts $x^2+4y^2-5z^2=1$ in a conic whose centre is at the point (2, 3, 4).

Ans. $x+6y-10z+20=0$.

Ex. 2. The locus of the centres of parallel plane sections of a conicoid is a diameter.

Ex. 3. The line joining a point P to the centre of a conicoid passes through the centre of the section of the conicoid by the polar plane of P.

Ex. 4. The centres of sections of a central conicoid that are parallel to a given line lie on a fixed plane.

Ex. 5. The centres of sections that pass through a given line lie on a conic.

Ex. 6. The centres of sections that pass through a given point lie on a conicoid.

Ex. 7. Find the locus of centres of sections of $ax^2+by^2+cz^2=1$ which touch $\alpha x^2+\beta y^2+\gamma z^2=1$.

Ans. $(ax^2+by^2+cz^2)^2=\frac{a^2x^2}{\alpha}+\frac{b^2y^2}{\beta}+\frac{c^2z^2}{\gamma}$.

72. Locus of mid-points of a system of parallel chords. It follows from equations (1) and (2) of § 71 that the mid-points of chords which are parallel to a fixed line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

lie in the plane $\quad alx+bmy+cnz=0.$

This is therefore the diametral plane which bisects the parallel chords (fig. 34).

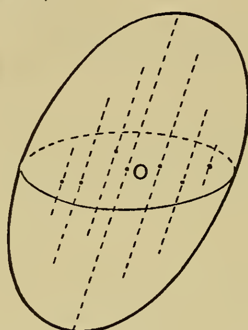

FIG. 34.

Compare the equation to the locus of the mid-points of parallel chords of the ellipse $ax^2+by^2=1$.

Ex. 1. Find the locus of the mid-points of chords of the conicoid $ax^2+by^2+cz^2=1$ which pass through the point (f, g, h).

Ans. $ax(x-f)+by(y-g)+cz(z-h)=0$.

Ex. 2. Prove that the mid-points of chords of $ax^2+by^2+cz^2=1$ which are parallel to $x=0$ and touch $x^2+y^2+z^2=r^2$ lie on the surface

$$by^2(bx^2+by^2+cz^2-br^2)+cz^2(cx^2+by^2+cz^2-cr^2)=0.$$

73. The locus of the tangents drawn from a given point. When the secant APQ, (fig. 32), becomes a tangent, P, Q, R coincide at the point of contact, and hence the points of contact of all the tangents from A lie on the polar plane of A, and therefore on the conic in which that plane cuts the surface. The locus of the tangents from A is therefore the cone generated by lines which pass through A and intersect

the conic in which the polar plane of A cuts the conicoid. This cone is the **enveloping cone** whose vertex is A. We may find its equation as follows: If A is (α, β, γ), and the line APQ, whose equations are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n},$$

meets the surface in coincident points, the equation (1) of § 36 has equal roots, and therefore

$$(al^2+bm^2+cn^2)(a\alpha^2+b\beta^2+c\gamma^2-1)$$
$$=(a\alpha l+b\beta m+c\gamma n)^2. \quad \text{.........(1)}$$

The locus of APQ is therefore the cone whose equation is

$$[a(x-\alpha)^2+b(y-\beta)^2+c(z-\gamma)^2][a\alpha^2+b\beta^2+c\gamma^2-1]$$
$$=[a\alpha(x-\alpha)+b\beta(y-\beta)+c\gamma(z-\gamma)]^2.$$

If $\quad \mathrm{S}\equiv ax^2+by^2+cz^2-1, \quad \mathrm{S}_1\equiv a\alpha^2+b\beta^2+c\gamma^2-1,$

and $\quad \mathrm{P}\equiv a\alpha x+b\beta y+c\gamma z-1,$

this equation may be written

$$(\mathrm{S}-2\mathrm{P}+\mathrm{S}_1)\mathrm{S}_1=(\mathrm{P}-\mathrm{S}_1)^2, \text{ i.e. } \mathrm{SS}_1=\mathrm{P}^2, \text{ or}$$

$$(ax^2+by^2+cz^2-1)(a\alpha^2+b\beta^2+c\gamma^2-1)=(a\alpha x+b\beta y+c\gamma z-1)^2.$$

Compare the equation to the pair of tangents from the point (α, β) to the ellipse $ax^2+by^2=1$.

Ex. 1. Find the locus of points from which three mutually perpendicular tangent lines can be drawn to the surface $ax^2+by^2+cz^2=1$.
Ans. $a(b+c)x^2+b(c+a)y^2+c(a+b)z^2=a+b+c.$

Ex. 2. Lines drawn from the centre of a central conicoid parallel to the generators of the enveloping cone whose vertex is A generate a cone which intersects the conicoid in two conics whose planes are parallel to the polar plane of A.

Ex. 3. Through a fixed point $(k, 0, 0)$ pairs of perpendicular tangent lines are drawn to the surface $ax^2+by^2+cz^2=1$. Shew that the plane through any pair touches the cone

$$\frac{(x-k)^2}{(ak^2-1)(b+c)}+\frac{y^2}{c(ak^2-1)-a}+\frac{z^2}{b(ak^2-1)-a}=0.$$

Ex. 4. The plane $z=a$ meets any enveloping cone of the sphere $x^2+y^2+z^2=a^2$ in a conic which has a focus at the point $(0, 0, a)$.

Ex. 5. Find the locus of a luminous point if the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ casts a circular shadow on the plane $z=0$.

Ans. $x=0,\ \dfrac{y^2}{b^2-a^2}+\dfrac{z^2}{c^2}=1\ ;\quad y=0,\ \dfrac{x^2}{a^2-b^2}+\dfrac{z^2}{c^2}=1.$

Ex. 6. If $S=0$, $u=0$, $v=0$ are the equations to a conicoid and two planes, prove that $S+\lambda uv=0$ represents a conicoid which passes through the conics in which the given planes cut the given conicoid, and interpret the equation $S+\lambda u^2=0$.

Ex. 7. Prove that if a straight line has three points on a conicoid, it lies wholly on the conicoid.

(The equation (1), § 66, is an identity.)

Ex. 8. A conicoid passes through a given point A and touches a given conicoid S at all points of the conic in which it is met by the polar plane of A. Prove that all the tangents from A to S lie on it. Hence find the equation to the enveloping cone of S whose vertex is A.

Ex. 9. The section of the enveloping cone of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ whose vertex is P by the plane $z=0$ is (i) a parabola, (ii) a rectangular hyperbola. Find the locus of P.

Ans. (i) $z=\pm c$, (ii) $\dfrac{x^2+y^2}{a^2+b^2}+\dfrac{z^2}{c^2}=1$.

74. The locus of the tangents which are parallel to a given line. Suppose that PQ is any chord and that M is its mid-point. Then if the line PQ moves parallel to itself till it meets the surface in coincident points, it becomes a tangent and M coincides with the point of contact. Therefore the point of contact of a tangent which is parallel

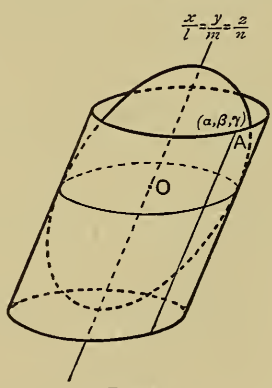

FIG. 35.

to a given line lies on the diametral plane which bisects all chords parallel to the line. This plane cuts the surface in a conic, and the locus of the tangents parallel to the given line is therefore the cylinder generated by the parallels to the given line which pass through the conic.

Let (α, β, γ) (fig. 35) be any point on a tangent parallel to a given line $x/l = y/m = z/n$.

Then since, by § 73 (1), the line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

touches the surface if

$$(al^2+bm^2+cn^2)(a\alpha^2+b\beta^2+c\gamma^2-1) = (a\alpha l+b\beta m+c\gamma n)^2,$$

the locus of (α, β, γ) is given by

$$(al^2+bm^2+cn^2)(ax^2+by^2+cz^2-1) = (alx+bmy+cnz)^2.$$

This equation therefore represents the **enveloping cylinder**, which is the locus of the tangents.

The enveloping cylinder may be considered to be a limiting case of the enveloping cone whose vertex is the point P, (lr, mr, nr) on the line $x/l = y/m = z/n$, as r tends to infinity. By § 73, the equation to the cone is

$$(ax^2+by^2+cz^2-1)\left(al^2+bm^2+cn^2-\frac{1}{r^2}\right) = \left(alx+bmy+cnz-\frac{1}{r}\right)^2,$$

whence the equation to the cylinder can be at once deduced.

Ex. 1. Prove that the enveloping cylinders of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, whose generators are parallel to the lines

$$\frac{x}{0} = \frac{y}{\pm\sqrt{a^2-b^2}} = \frac{z}{c},$$

meet the plane $z=0$ in circles.

Ex. 2. Prove that the polar of a line AB is the line of intersection of the planes of contact of the enveloping cone whose vertex is A and the enveloping cylinder whose generators are parallel to AB.

75. Normals. In discussing the properties of the normals we shall confine our attention to the normals to the ellipsoid, the most familiar of the central conicoids.

Consider the ellipsoid whose equation, referred to rectangular axes, is $x^2/a^2+y^2/b^2+z^2/c^2=1$. If the plane

$$p = x\cos\alpha + y\cos\beta + z\cos\gamma, \quad (p>0),$$

is a tangent plane whose point of contact is (x', y', z'), we have, as in § 68,

$$\cos\alpha = \frac{px'}{a^2}, \quad \cos\beta = \frac{py'}{b^2}, \quad \cos\gamma = \frac{pz'}{c^2};$$

that is, the direction-cosines of the outward-drawn normal at (x', y', z') are $\dfrac{px'}{a^2}, \dfrac{py'}{b^2}, \dfrac{pz'}{c^2}$, where p is the perpendicular from the centre to the tangent plane at the point. The equations to the normal at (x', y', z') are therefore

$$\frac{x-x'}{\dfrac{px'}{a^2}}=\frac{y-y'}{\dfrac{py'}{b^2}}=\frac{z-z'}{\dfrac{pz'}{c^2}} \quad (=r).$$

Ex. 1. If the normal at P meets the principal planes in G_1, G_2, G_3, shew that

$$PG_1 : PG_2 : PG_3 = a^2 : b^2 : c^2.$$

Putting 0 for x in the equations to the normal, we obtain

$$r = PG_1 = -\frac{a^2}{p}, \text{ etc.}$$

Ex. 2. If $PG_1^2+PG_2^2+PG_3^2=k^2$, find the locus of P.

Ans. The curve of intersection of the given ellipsoid and the ellipsoid $\dfrac{x^2}{a^4}+\dfrac{y^2}{b^4}+\dfrac{z^2}{c^4}=\dfrac{k^2}{a^4+b^4+c^4}$.

Ex. 3. Find the length of the normal chord through P, and prove that if it is equal to $4PG_3$, P lies on the cone

$$\frac{x^2}{a^6}(2c^2-a^2)+\frac{y^2}{b^6}(2c^2-b^2)+\frac{z^2}{c^4}=0.$$

Ans. $2\Big/\left(p^3\Sigma\dfrac{x'^2}{a^6}\right)$.

Ex. 4. The normal at a variable point P meets the plane XOY in A, and AQ is drawn parallel to OZ and equal to AP. Prove that the locus of Q is given by

$$\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}+\frac{z^2}{c^2}=1.$$

Find the locus of R if OR is drawn from the centre equal and parallel to AP. *Ans.* $a^2x^2+b^2y^2+c^2z^2=c^4$.

Ex. 5. If the normals at P and Q, points on the ellipsoid, intersect, PQ is at right angles to its polar with respect to the ellipsoid.

76. The normals from a given point. If the normal at (x', y', z') passes through a given point (α, β, γ), then

$$\frac{\alpha-x'}{\dfrac{x'}{a^2}}=\frac{\beta-y'}{\dfrac{y'}{b^2}}=\frac{\gamma-z'}{\dfrac{z'}{c^2}}, \quad \text{.................(1)}$$

and if each of these fractions is equal to λ,

$$x'=\frac{a^2\alpha}{a^2+\lambda}, \quad y'=\frac{b^2\beta}{b^2+\lambda}, \quad z'=\frac{c^2\gamma}{c^2+\lambda}. \quad \text{..........(2)}$$

Therefore, since (x', y', z') is on the ellipsoid,

$$\frac{a^2\alpha^2}{(a^2+\lambda)^2}+\frac{b^2\beta^2}{(b^2+\lambda)^2}+\frac{c^2\gamma^2}{(c^2+\lambda)^2}=1. \qquad \text{.............. (3)}$$

This equation gives six values of λ, to each of which corresponds a point (x', y', z'), and therefore there are six points on the ellipsoid the normals at which pass through (α, β, γ).

Ex. 1. Prove that equation (3) gives at least two real values of λ. (If $F(\lambda)\equiv(\lambda+a^2)^2(\lambda+b^2)^2(\lambda+c^2)^2-\Sigma a^2\alpha^2(\lambda+b^2)^2(\lambda+c^2)^2$, $F(\lambda)$ is negative when $\lambda=-a^2, -b^2, -c^2$, and is positive when $\lambda=\pm\infty$.)

Ex. 2. Prove that four normals to the ellipsoid pass through any point of the curve of intersection of the ellipsoid and the conicoid

$$x^2(b^2+c^2)+y^2(c^2+a^2)+z^2(a^2+b^2)=b^2c^2+c^2a^2+a^2b^2.$$

It follows from equations (1) that the feet of the normals from (α, β, γ) to the ellipsoid lie on the three cylinders

$$b^2z(\beta-y)=c^2y(\gamma-z), \quad c^2x(\gamma-z)=a^2z(\alpha-x),$$
$$a^2y(\alpha-x)=b^2x(\beta-y).$$

Compare the equation to the rectangular hyperbola through the feet of the normals from the point (α, β) to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

These cylinders have a common curve of intersection, and equations (2) express the coordinates of any point on it in terms of a parameter λ. The points where the curve meets a given plane

$$ux+vy+wz+d=0$$

are given by $$\frac{ua^2\alpha}{a^2+\lambda}+\frac{vb^2\beta}{b^2+\lambda}+\frac{wc^2\gamma}{c^2+\lambda}+d=0,$$

and as this determines three values of λ, the plane meets the curve in three points, and the curve is therefore a *cubic curve.* The feet of the normals from (α, β, γ) to the ellipsoid are therefore the six points of intersection of the ellipsoid and a certain cubic curve.

If the normal at (x', y', z') passes through (α, β, γ) and has direction-cosines l, m, n,

$$l=\frac{px'}{a^2}=\frac{p\alpha}{a^2+\lambda}, \quad m=\frac{p\beta}{b^2+\lambda}, \quad n=\frac{p\gamma}{c^2+\lambda},$$

and therefore

$$\frac{\alpha}{l}(b^2-c^2)+\frac{\beta}{m}(c^2-a^2)+\frac{\gamma}{n}(a^2-b^2)=0.$$

This shews that the normal whose equations are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

is a generator of the cone

$$\frac{\alpha(b^2-c^2)}{x-\alpha}+\frac{\beta(c^2-a^2)}{y-\beta}+\frac{\gamma(a^2-b^2)}{z-\gamma}=0.$$

Hence the six normals from (α, β, γ) lie on a cone of the second degree.

Ex. 3. If P is the point (α, β, γ), prove that the line PO, the parallels through P to the axes, and the perpendicular from P to its polar plane, lie on the cone.

Ex. 4. Shew that the cubic curve lies on the cone.

Ex. 5. Prove that the feet of the six normals from (α, β, γ) lie on the curve of intersection of the ellipsoid and the cone

$$\frac{a^2(b^2-c^2)\alpha}{x}+\frac{b^2(c^2-a^2)\beta}{y}+\frac{c^2(a^2-b^2)\gamma}{z}=0.$$

Ex. 6. The generators of the cone which contains the normals from a given point to an ellipsoid are at right angles to their polars with respect to the ellipsoid.

Ex. 7. A is a fixed point and P a variable point such that its polar plane is at right angles to AP. Shew that the locus of P is the cubic curve through the feet of the normals from A.

Ex. 8. If P, Q, R ; P′, Q′, R′ are the feet of the six normals from a point to the ellipsoid, and the plane PQR is given by $lx+my+nz=p$, then the plane P′Q′R′ is given by

$$\frac{x}{a^2l}+\frac{y}{b^2m}+\frac{z}{c^2n}+\frac{1}{p}=0.$$

(If P′Q′R′ is given by $l'x+m'y+n'z=p'$, then

$$x^2/a^2+y^2/b^2+z^2/c^2-1+\lambda(lx+my+nz-p)(l'x+m'y+n'z-p')=0$$

represents the cone containing the normals, for some value of λ.)

Ex. 9. If A, A′ are the poles of the planes PQR, P′Q′R′,

$$\mathrm{AA'^2-OA^2-OA'^2}=2(a^2+b^2+c^2).$$

77. Conjugate diameters and conjugate diametral planes of the ellipsoid. If the equation to the ellipsoid

is $x^2/a^2+y^2/b^2+z^2/c^2=1$, the axes are conjugate diameters and the coordinate planes are conjugate diametral planes, (§ 65). If P, (x_1, y_1, z_1) is any point on the ellipsoid, the diametral plane of OP has for its equation, (§ 72),

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=0.$$

Let Q, (x_2, y_2, z_2) be any point on this plane and on the ellipsoid, then

$$\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+\frac{z_1z_2}{c^2}=0.$$

Hence, if Q is on the diametral plane of OP, P is on the diametral plane of OQ.

If the diametral planes of OP and OQ intersect in the diameter OR, (fig. 36), R is on the diametral planes of OP and OQ, and therefore P and Q are on the diametral plane of OR; that is, the diametral plane of OR is the plane OPQ. Thus the planes QOR, ROP, POQ are the diametral planes of OP, OQ, OR respectively, and they are therefore conjugate diametral planes, and OP, OQ, OR are conjugate diameters.

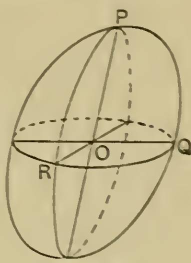

FIG. 36.

If R is (x_3, y_3, z_3), we have

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1,\quad \frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}+\frac{z_2^2}{c^2}=1,\quad \frac{x_3^2}{a^2}+\frac{y_3^2}{b^2}+\frac{z_3^2}{c^2}=1, \quad (\text{A}')$$

$$\frac{x_2x_3}{a^2}+\frac{y_2y_3}{b^2}+\frac{z_2z_3}{c^2}=0,\quad \frac{x_3x_1}{a^2}+\frac{y_3y_1}{b^2}+\frac{z_3z_1}{c^2}=0,$$

$$\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+\frac{z_1z_2}{c^2}=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots(\text{B}')$$

These correspond exactly to equations (A) and (B) of § 53, and shew that

$$\frac{x_1}{a},\ \frac{y_1}{b},\ \frac{z_1}{c};\quad \frac{x_2}{a},\ \frac{y_2}{b},\ \frac{z_2}{c};\quad \frac{x_3}{a},\ \frac{y_3}{b},\ \frac{z_3}{c}$$

are the direction-cosines of three mutually perpendicular

lines referred to rectangular axes. Therefore, as in § 53, we deduce,

$$\left.\begin{aligned} x_1^2+x_2^2+x_3^2&=a^2,\\ y_1^2+y_2^2+y_3^2&=b^2,\\ z_1^2+z_2^2+z_3^2&=c^2;\end{aligned}\right\}\text{......(C}'\text{)}\qquad \left.\begin{aligned} y_1z_1+y_2z_2+y_3z_3&=0,\\ z_1x_1+z_2x_2+z_3x_3&=0,\\ x_1y_1+x_2y_2+x_3y_3&=0;\end{aligned}\right\}\text{......(D}'\text{)}$$

$$\frac{x_1}{a}=\pm\frac{(y_2z_3-z_2y_3)}{bc},\quad \frac{y_1}{b}=\pm\frac{(z_2x_3-x_2z_3)}{ca},\quad \frac{z_1}{c}=\pm\frac{(x_2y_3-y_2x_3)}{ab},$$

$$\frac{x_2}{a}=\pm\frac{(y_3z_1-z_3y_1)}{bc},\ \text{etc., etc.;} \ \ldots\ldots\ldots\ldots\text{(E}'\text{)}$$

$$\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}=\pm abc.$$

If the axes to which the ellipsoid is referred are rectangular, equations (C′) give, on adding,

$$OP^2+OQ^2+OR^2=a^2+b^2+c^2.$$

Hence the sum of the squares on any three conjugate semi-diameters is constant. From the last equation we deduce that the volume of the parallelepiped which has OP, OQ, OR for coterminous edges is constant and equal to abc. Again, if A_1, A_2, A_3 are the areas QOR, ROP, POQ and l_r, m_r, n_r, $(r=1,\ 2,\ 3)$, are the direction-cosines of the normals to the planes QOR, ROP, POQ, projecting A_1 on the plane $x=0$, we obtain

$$l_1A_1=\frac{y_2z_3-z_2y_3}{2}=\pm\frac{bcx_1}{2a},\quad \text{by (E}'\text{);}$$

similarly, $$m_1A_1=\pm\frac{cay_1}{2b},\quad n_1A_1=\pm\frac{abz_1}{2c};$$

$$l_2A_2=\pm\frac{bcx_2}{2a},\quad m_2A_2=\pm\frac{cay_2}{2b},\quad n_2A_2=\pm\frac{abz_2}{2c};$$

$$l_3A_3=\pm\frac{bcx_3}{2a},\quad m_3A_3=\pm\frac{cay_3}{2b},\quad n_3A_3=\pm\frac{abz_3}{2c}.$$

Therefore, squaring and adding, we have, by (C′),

$$A_1^2+A_2^2+A_3^2=\tfrac{1}{4}(b^2c^2+c^2a^2+a^2b^2).$$

Ex. 1. Find the equation to the plane **PQR**.

If the equation is $lx+my+nz=p$, then $lx_1+my_1+nz_1=p$,

$$lx_2+my_2+nz_2=p, \quad lx_3+my_3+nz_3=p.$$

Multiply by x_1, x_2, x_3 respectively, and add; then by (c′) and (D′),

$$la^2=p(x_1+x_2+x_3), \text{ etc.}$$

The required equation is therefore

$$\frac{x(x_1+x_2+x_3)}{a^2}+\frac{y(y_1+y_2+y_3)}{b^2}+\frac{z(z_1+z_2+z_3)}{c^2}=1.$$

Ex. 2. Shew that the plane **PQR** touches the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{1}{3}$ at the centroid of the triangle **PQR**.

Ex. 3. Prove that the pole of the plane **PQR** lies on the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=3.$$

Ex. 4. The locus of the foot of the perpendicular from the centre to the plane through the extremities of three conjugate semi-diameters is $a^2x^2+b^2y^2+c^2z^2=3(x^2+y^2+z^2)^2$.

Ex. 5. Prove that the sum of the squares of the projections of **OP, OQ, OR**, (i) on any line, (ii) on any plane, is constant.

Ex. 6. Shew that any two sets of conjugate diameters lie on a cone of the second degree. (Cf. § 59, Ex. 6.)

Ex. 7. Shew that any two sets of conjugate diametral planes touch a cone of the second degree. (Apply § 61, Ex. 4.)

Ex. 8. If the axes are rectangular, find the locus of the equal conjugate diameters of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$.

If r is the length of one of the equal conjugate diameters,

$$3r^2=a^2+b^2+c^2,$$

and
$$\frac{l^2+m^2+n^2}{r^2}=\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2},$$

where l, m, n are the direction-cosines. Therefore the diameter is a generator of the cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{3(x^2+y^2+z^2)}{a^2+b^2+c^2},$$

or
$$\frac{x^2}{a^2}(2a^2-b^2-c^2)+\frac{y^2}{b^2}(2b^2-c^2-a^2)+\frac{z^2}{c^2}(2c^2-a^2-b^2)=0.$$

Ex. 9. Shew that the plane through a pair of equal conjugate diameters touches the cone $\Sigma\frac{x^2}{a^2(2a^2-b^2-c^2)}=0$.

Ex. 10. If λ, μ, ν are the angles between a set of equal conjugate diameters,

$$\cos^2\lambda+\cos^2\mu+\cos^2\nu=\frac{3\Sigma(b^2-c^2)^2}{2(a^2+b^2+c^2)^2}.$$

Ex. 11. If **OP, OQ, OR** are equal conjugate diameters, and **S** is the pole of the plane **PQR**, the tetrahedron **SPQR** has any pair of opposite edges at right angles.

Ex. 12. If OP, OQ, OR are conjugate diameters and p_1, p_2, p_3; π_1, π_2, π_3 are their projections on any two given lines, $p_1\pi_1+p_2\pi_2+p_3\pi_3$ is constant.

Ex. 13. If, through a given point, chords are drawn parallel to OP, OQ, OR, the sum of the squares of the ratios of the respective chords to OP, OQ, OR is constant.

Ex. 14. The locus of the point of intersection of three tangent planes to $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, which are parallel to conjugate diametral planes of $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1$, is $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}$. What does this theorem become when $\alpha=\beta=\gamma$?

Ex. 15. Shew that conjugate diameters satisfy the condition of Ex. 8, § 70, for conjugate lines.

Since the plane POQ, (fig. 36), bisects all chords of the conicoid which are parallel to OR, the line OQ bisects all chords of the conic ROQ which are parallel to OR. Similarly OR bisects all chords of the conic which are parallel to OQ; and therefore OR and OQ are conjugate diameters of the ellipse ROQ. But Q is any point on the ellipse; therefore OP and any pair of conjugate diameters of the ellipse in which the diametral plane of OP cuts the ellipsoid are conjugate diameters of the ellipsoid.

Ex. 16. P is any point on the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and 2α and 2β are the principal axes of the section of the ellipsoid by the diametral plane of OP. Prove that $OP^2=a^2+b^2+c^2-\alpha^2-\beta^2$, and that $\alpha\beta p=abc$, where p is the perpendicular from O to the tangent plane at P.

Ex. 17. If 2α and 2β are the principal axes of the section of the ellipsoid by the plane $lx+my+nz=0$, prove that

$$\alpha^2+\beta^2=\frac{a^2(b^2+c^2)l^2+b^2(c^2+a^2)m^2+c^2(a^2+b^2)n^2}{a^2l^2+b^2m^2+c^2n^2},$$

$$\alpha^2\beta^2=\frac{a^2b^2c^2(l^2+m^2+n^2)}{a^2l^2+b^2m^2+c^2n^2}.$$

Ex. 18. If P, (x_1, y_1, z_1) is a point on the ellipsoid and (ξ_1, η_1, ζ_1), (ξ_2, η_2, ζ_2) are extremities of the principal axes of the section of the ellipsoid by the diametral plane of OP, prove that

$$\frac{\xi_1\xi_2}{a^2(b^2-c^2)}=\frac{\eta_1\eta_2}{b^2(c^2-a^2)}=\frac{\zeta_1\zeta_2}{c^2(a^2-b^2)},$$

$$(b^2-c^2)\frac{x_1}{\xi_1}+(c^2-a^2)\frac{y_1}{\eta_1}+(a^2-b^2)\frac{z_1}{\zeta_1}=0.$$

Conjugate diameters of the hyperboloids. The equation of a hyperboloid of one sheet referred to three conjugate diameters as coordinate axes is $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}-\frac{z^2}{\gamma^2}=1$. Hence it appears that the x- and y-axes meet the surface in real points $(\pm\alpha, 0, 0)$, $(0, \pm\beta, 0)$, and that the z-axis does not intersect the surface. The z-axis, however, intersects the hyperboloid of two sheets whose equation is $-\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1$ at the points $(0, 0, \pm\gamma)$, and these points are taken as the extremities of the third of the three conjugate diameters.

Hence, if P, (x_1, y_1, z_1), Q, (x_2, y_2, z_2), R, (x_3, y_3, z_3) are the extremities of a set of conjugate semi-diameters of the hyperboloid of one sheet,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

it follows, as for the ellipsoid, that

$$x_1^2+x_2^2-x_3^2=a^2, \quad y_1^2+y_2^2-y_3^2=b^2, \quad z_1^2+z_2^2-z_3^2=-c^2, \text{ etc.};$$

and therefore, that if the axes are rectangular,

$$\mathrm{OP}^2+\mathrm{OQ}^2-\mathrm{OR}^2=a^2+b^2-c^2$$

and
$$\mathrm{A}_1^2+\mathrm{A}_2^2-\mathrm{A}_3^2=\tfrac{1}{4}(b^2c^2+c^2a^2-a^2b^2).$$

Similarly, if one of a set of three conjugate diameters of the hyperboloid of two sheets, $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, intersects the surface, the other two do not, but they intersect the hyperboloid of one sheet,

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

and the points of intersection are taken as their extremities. Hence if P, Q, R are the extremities, and the axes are rectangular, we have

$$\mathrm{OP}^2-\mathrm{OQ}^2-\mathrm{OR}^2=a^2-b^2-c^2$$

and
$$\mathrm{A}_1^2-\mathrm{A}_2^2-\mathrm{A}_3^2=\tfrac{1}{4}(b^2c^2-c^2a^2-a^2b^2).$$

THE CONE.

78. A homogeneous equation of the form

$$ax^2+by^2+cz^2=0$$

represents a cone. If (x', y', z') is any point on the cone, $(-x', -y', -z')$ lies also on the cone, and therefore we may consider the cone as a central surface, the vertex being the centre. The coordinate planes are conjugate diametral planes and the coordinate axes are conjugate diameters.

We easily find, as in the case of the other central conicoids, the following results:

The tangent plane at (x', y', z') has for its equation

$$axx'+byy'+czz'=0.$$

The plane $lx+my+nz=0$ touches the cone if

$$l^2/a+m^2/b+n^2/c=0.$$

The polar plane of P, (α, β, γ) is given by

$$a\alpha x+b\beta y+c\gamma z=0.$$

The section whose centre is at (α, β, γ) has the equation

$$(x-\alpha)a\alpha+(y-\beta)b\beta+(z-\gamma)c\gamma=0.$$

The diametral plane of the line $x/l=y/m=z/n$ is

$$alx+bmy+cnz=0.$$

The locus of the tangents drawn from P, (α, β, γ) is the pair of tangent planes whose line of intersection is OP. They are given by

$$(ax^2+by^2+cz^2)(a\alpha^2+b\beta^2+c\gamma^2)=(a\alpha x+b\beta y+c\gamma z)^2.$$

The diametral plane of OP is also the polar plane of P.

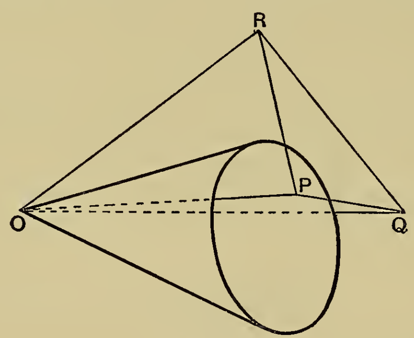

FIG. 37.

Conjugate diameters. Let OP, OQ, OR, (fig. 37), any three conjugate diameters of the cone, meet any plane in P, Q, R. The plane meets the cone in a conic, and QR is the locus of the harmonic conjugates of P with respect to the points in which any secant through P cuts the conic; *i.e.* QR is the polar of P with respect to the conic. Similarly, RP and PQ are the polars of Q and R, and therefore the triangle PQR is self-polar with respect to the conic. Conversely, if PQR

is any triangle self-polar with respect to the conic in which the plane PQR cuts the cone, OP, OQ, OR are conjugate diameters of the cone. For the polar plane of P passes through the line QR and through the vertex, and therefore OQR is the polar plane of P, or the diametral plane of OP; and similarly, ORP, OPQ are the diametral planes of OQ and OR.

Ex. 1. The locus of the asymptotic lines drawn from O to the conicoid $ax^2+by^2+cz^2=1$ is the asymptotic cone $ax^2+by^2+cz^2=0$.

Ex. 2. The hyperboloids

$$x^2/a^2+y^2/b^2-z^2/c^2=1, \quad -x^2/a^2-y^2/b^2+z^2/c^2=1$$

have the same asymptotic cone. Draw a figure shewing the cone and the two hyperboloids.

Ex. 3. The section of a hyperboloid by a plane which is parallel to a tangent plane of the asymptotic cone is a parabola.

Ex. 4. If a plane through the origin cuts the cones

$$ax^2+by^2+cz^2=0, \quad \alpha x^2+\beta y^2+\gamma z^2=0$$

in lines which form a harmonic pencil, it touches the cone

$$\frac{x^2}{b\gamma+c\beta}+\frac{y^2}{c\alpha+a\gamma}+\frac{z^2}{a\beta+b\alpha}=0.$$

For the following examples the axes are rectangular.

Ex. 5. Planes which cut $ax^2+by^2+cz^2=0$ in perpendicular generators touch

$$\frac{x^2}{b+c}+\frac{y^2}{c+a}+\frac{z^2}{a+b}=0.$$

Ex. 6. The lines of intersection of pairs of tangent planes to $ax^2+by^2+cz^2=0$ which touch along perpendicular generators lie on the cone

$$a^2(b+c)x^2+b^2(c+a)y^2+c^2(a+b)z^2=0.$$

Ex. 7. Perpendicular tangent planes to $ax^2+by^2+cz^2=0$ intersect in generators of the cone

$$a(b+c)x^2+b(c+a)y^2+c(a+b)z^2=0.$$

Ex. 8. If the cone $\mathrm{A}x^2+\mathrm{B}y^2+\mathrm{C}z^2+2\mathrm{F}yz+2\mathrm{G}zx+2\mathrm{H}xy=0$ passes through a set of conjugate diameters of the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1, \quad \text{then} \quad \mathrm{A}a^2+\mathrm{B}b^2+\mathrm{C}c^2=0.$$

Ex. 9. If three conjugate diameters of an ellipsoid meet the director sphere in P, Q, R, the plane PQR touches the ellipsoid.

Ex. 10. Find the equation to the normal plane (*i.e.* at right angles to the tangent plane) of the cone $ax^2+by^2+cz^2=0$ which passes through the generator $x/l=y/m=z/n$.

Ans. $\Sigma\dfrac{(b-c)x}{l}=0.$

Ex. 11. Lines drawn through the origin at right angles to normal planes of the cone $ax^2+by^2+cz^2=0$ generate the cone

$$\frac{a(b-c)^2}{x^2}+\frac{b(c-a)^2}{y^2}+\frac{c(a-b)^2}{z^2}=0.$$

Ex. 12. If the two cones $ax^2+by^2+cz^2=0$, $\alpha x^2+\beta y^2+\gamma z^2=0$ have each sets of three mutually perpendicular generators, any two planes which pass through their four common generators are at right angles.

THE PARABOLOIDS.

79. The locus of the equation

$$(1)\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c}, \qquad (2)\quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2z}{c}.$$

The equation (1) represents the surface generated by the variable ellipse $z=k$, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2k}{c}$. This ellipse is imaginary unless k and c have the same sign, hence the centre of the

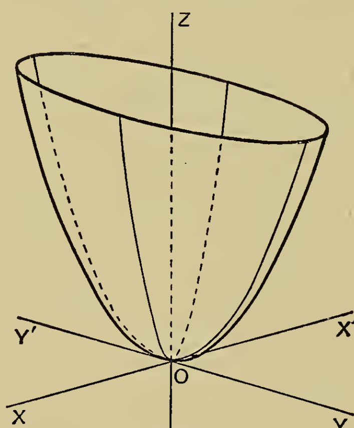

FIG. 38.

ellipse lies on OZ if $c>0$ and on OZ′ if $c<0$. The sections of the surface by planes parallel to the coordinate planes YOZ, ZOX are parabolas. Fig. 38 shews the form and position of the surface for a positive value of c. The surface is the **elliptic paraboloid**.

The equation (2) represents the surface generated by the variable hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2k}{c}$, $z=k$. The hyperbola is real for all real values of k, and its centre passes in turn through every point on Z′Z. When $k=0$ the hyperbola degenerates into the two lines $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0, z=0$. The sections of the surface by the planes $z=k$, $z=-k$ project on the

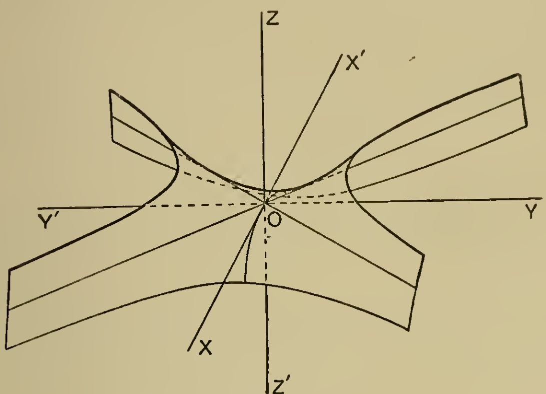

FIG. 39.

plane XOY into conjugate hyperbolas whose asymptotes are $z=0$, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$. The sections by planes parallel to YOZ, ZOX are parabolas. The surface is the **hyperbolic paraboloid**, and fig. 39 shews the form and position of the surface for a negative value of c.

80. Conjugate diametral planes. An equation of the form

$$ax^2+by^2=2z$$

represents a paraboloid. Any line in the plane XOY which passes through the origin meets the surface in two coincident points, and hence the plane XOY is the tangent plane at the origin. The planes YOZ, ZOX bisect chords parallel to OX and OY respectively. Each is therefore parallel to the chords bisected by the other. Such pairs of planes are called **conjugate diametral planes** of the paraboloid.

81. Diameters. If A is the point (α, β, γ), and the equations to a line through A are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad (=r),$$

the distances from A to the points of intersection of the line and the paraboloid are given by

$$r^2(al^2+bm^2)+2r(a\alpha l+b\beta m-n)+a\alpha^2+b\beta^2-2\gamma=0. \quad \ldots(1)$$

If $l=m=0$, one value of r is infinite, and therefore a line parallel to the z-axis meets the paraboloid in one point at an infinite distance, and in a point P whose distance from A is given by

$$r=\frac{a\alpha^2+b\beta^2-2\gamma}{-2(a\alpha l+b\beta m-n)}=\frac{a\alpha^2+b\beta^2-2\gamma}{2n}.$$

Such a line is called a **diameter**, and P is the extremity of the diameter.

Hence $ax^2+by^2=2z$ represents a paraboloid, referred to a tangent plane, and two conjugate diametral planes through the point of contact, as coordinate planes. One of the coordinate axes is the diameter through the point of contact. If the axes are rectangular, so that the tangent plane at O is at right angles to the diameter through O, O is the **vertex** of the paraboloid, and the diameter through O is the **axis.** The coordinate planes YOZ, ZOX are then **principal planes.**

Ex. What surface is represented by the equation $xy=2cz$?

82. Tangent planes. We find, as in § 67, the equation to the tangent plane at the point (α, β, γ) on the paraboloid,

$$a\alpha x+b\beta y=z+\gamma.$$

If $lx+my+nz=p$ is a tangent plane and (α, β, γ) is the point of contact,

$$\alpha=\frac{-l}{an}, \quad \beta=\frac{-m}{bn}, \quad \gamma=\frac{-p}{n},$$

and therefore $$\frac{l^2}{a}+\frac{m^2}{b}+2np=0.$$

Hence $2n(lx+my+nz)+\frac{l^2}{a}+\frac{m^2}{b}=0$ is the equation to the tangent plane to the paraboloid which is parallel to the plane $lx+my+nz=0$.

If the axes are rectangular and

$$2n_r(l_rx+m_ry+n_rz)+\frac{l_r^2}{a}+\frac{m_r^2}{b}=0, \quad (r=1, 2, 3),$$

represent three mutually perpendicular tangent planes, we have, by addition,

$$2z+\frac{1}{a}+\frac{1}{b}=0,$$

and therefore the locus of the point of intersection of three mutually perpendicular tangent planes is a plane at right angles to the axis of the paraboloid.

Ex. 1. Shew that the plane $8x-6y-z=5$ touches the paraboloid $\frac{x^2}{2}-\frac{y^2}{3}=z$, and find the coordinates of the point of contact.

Ans. (8, 9, 5).

Ex. 2. Two perpendicular tangent planes to the paraboloid $\frac{x^2}{a}+\frac{y^2}{b}=2z$ intersect in a straight line lying in the plane $x=0$. Shew that the line touches the parabola

$$x=0, \quad y^2=(a+b)(2z+a).$$

Ex. 3. Shew that the locus of the tangents from a point (α, β, γ) to the paraboloid $ax^2+by^2=2z$ is given by

$$(ax^2+by^2-2z)(a\alpha^2+b\beta^2-2\gamma)=(a\alpha x+b\beta y-z-\gamma)^2.$$

Ex. 4. Find the locus of points from which three mutually perpendicular tangents can be drawn to the paraboloid.

Ans. $ab(x^2+y^2)-2(a+b)z-1=0$.

83. Diametral planes. If a line OP has equations $x/l=y/m=z/n$, the diametral plane of OP, *i.e.* the locus of the mid-points of chords parallel to OP, is given by $alx+bmy-n=0$. Hence all diametral planes are parallel to the axis of the paraboloid, and conversely any plane parallel to the axis is a diametral plane. If OQ, whose equations are $x/l'=y/m'=z/n'$, is parallel to the diametral plane of OP,

$$all'+bmm'=0. \qquad \text{.......................(1)}$$

Hence OP is parallel to the diametral plane of OQ, and the diametral planes of OP and OQ are conjugate.

Equation (1) is the condition that the lines $alx+bmy=0$, $al'x+bm'y=0$, in the plane $z=k$, should be conjugate diameters of the conic $ax^2+by^2=2k$. Hence any plane meets a pair of conjugate diametral planes of a paraboloid in lines which are parallel to conjugate diameters of the conic in which the plane meets the surface.

Ex. 1. The locus of the centres of a system of parallel plane sections of a paraboloid is a diameter.

Ex. 2. The plane $3x+4y=1$ is a diametral plane of the paraboloid $5x^2+6y^2=2z$. Find the equations to the chord through (3, 4, 5) which it bisects.

Ans. $\dfrac{x-3}{9}=\dfrac{y-4}{10}=\dfrac{z-5}{15}$.

Ex. 3. Any diametral plane cuts the paraboloid in a parabola, and parallel diametral planes cut it in equal parabolas.

84. The normals. If $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=2z$ represents an elliptic paraboloid, referred to rectangular axes, the normal at (x', y', z') has for equations

$$\frac{x-x'}{\dfrac{x'}{a^2}}=\frac{y-y'}{\dfrac{y'}{b^2}}=\frac{z-z'}{-1}.$$

If this normal passes through a given point (α, β, γ),

$$\frac{\alpha-x'}{\dfrac{x'}{a^2}}=\frac{\beta-y'}{\dfrac{y'}{b^2}}=\frac{\gamma-z'}{-1}=\lambda, \quad \text{say.}$$

Therefore $\quad x'=\dfrac{a^2\alpha}{a^2+\lambda}, \quad y'=\dfrac{b^2\beta}{b^2+\lambda}, \quad z'=\gamma+\lambda,$

and
$$\frac{a^2\alpha^2}{(a^2+\lambda)^2}+\frac{b^2\beta^2}{(b^2+\lambda)^2}=2(\gamma+\lambda).$$

This equation gives five values of λ, and hence there are five points on the paraboloid the normals at which pass through a given point.

Ex. 1. Prove that the feet of the normals from any point to the paraboloid lie on a cubic curve.

Ex. 2. Prove that the normals from (α, β, γ) to the paraboloid lie on the cone

$$\frac{\alpha}{x-\alpha} - \frac{\beta}{y-\beta} + \frac{a^2-b^2}{z-\gamma} = 0.$$

Ex. 3. Prove that the cubic curve lies on this cone.

Ex. 4. Prove that the perpendicular from (α, β, γ) to its polar plane lies on the cone.

Ex. 5. In general three normals can be drawn from a given point to the paraboloid of revolution $x^2+y^2=2az$, but if the point lies on the surface $27a(x^2+y^2)+8(a-z)^3=0$, two of the three normals coincide.

Ex. 6. Shew that the feet of the normals from the point (α, β, γ) to the paraboloid $x^2+y^2=2az$ lie on the sphere

$$x^2+y^2+z^2-z(a+\gamma)-\frac{y}{2\beta}(\alpha^2+\beta^2)=0.$$

Ex. 7. Shew that the centre of the circle through the feet of the three normals from the point (α, β, γ) to the paraboloid $x^2+y^2=2az$ is

$$\left(\frac{\alpha}{4}, \frac{\beta}{4}, \frac{\gamma+a}{2}\right).$$

*Examples IV.

1. Two asymptotic lines can be drawn from a point P to a conicoid $ax^2+by^2+cz^2=1$, and they are at right angles if P lies on the cone

$$a^2(b+c)x^2+b^2(c+a)y^2+c^2(a+b)z^2=0.$$

2. The lines in which the plane $lx+my+nz=0$ cuts the cone $\alpha x^2+\beta y^2+\gamma z^2=0$ are conjugate diameters of the ellipse in which it cuts the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Prove that the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ lies on the cone $a^2(\beta b^2+\gamma c^2)x^2+b^2(\gamma c^2+\alpha a^2)y^2+c^2(\alpha a^2+\beta b^2)z^2=0.$

3. P and Q are points on an ellipsoid. The normal at P meets the tangent plane at Q in R; the normal at Q meets the tangent plane at P in S. If the perpendiculars from the centre to the tangent planes at P and Q are p_1, p_2, prove that PR : QS $=p_2:p_1$.

4. The line of intersection of the tangent planes at P and Q, points on $ax^2+by^2+cz^2=1$, passes through a fixed point A, (α, β, γ), and is parallel to the plane XOY. Shew that the locus of the mid-point of PQ is the conic in which the polar plane of A cuts the surface

$$ax^2+by^2+cz^2=z/\gamma.$$

5. Shew that the greatest value of the shortest distance between the axis of x and a normal to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ is $b-c$.

6. Plane sections of an ellipsoid which have their centres on a given straight line are parallel to a fixed straight line and touch a parabolic cylinder.

7. OP, OQ, OR are conjugate diameters of an ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

At Q and R tangent lines are drawn parallel to OP, and p_1, p_2 are their distances from O. The perpendicular from O to the tangent plane at right angles to OP is p. Prove that

$$p^2+p_1^2+p_2^2=a^2+b^2+c^2.$$

8. Conjugate diameters of

$$a_1x^2+b_1y^2+c_1z^2=1 \quad \text{meet} \quad a_2x^2+b_2y^2+c_2z^2=1$$

in P, Q, R. Shew that the plane PQR touches the conicoid

$$a_3x^2+b_3y^2+c_3z^2=1,$$

where

$$\frac{a_3}{a_1}=\frac{b_3}{b_1}=\frac{c_3}{c_1}=\frac{a_2}{a_1}+\frac{b_2}{b_1}+\frac{c_2}{c_1}.$$

9. The ellipsoid which has as conjugate diameters the three straight lines that bisect pairs of opposite edges of a tetrahedron touches the edges.

10. Shew that the projections of the normals to an ellipsoid at P, Q, R, the extremities of conjugate diameters on the plane PQR, are concurrent.

11. If through a fixed point P, (α, β, γ) on the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ perpendiculars are drawn to any three conjugate diameters, the plane through the feet of the perpendiculars passes through the fixed point

$$\left(\frac{a^2\alpha}{a^2+b^2+c^2}, \quad \frac{b^2\beta}{a^2+b^2+c^2}, \quad \frac{c^2\gamma}{a^2+b^2+c^2}\right).$$

12. If perpendiculars be drawn from any point P on the ellipsoid to any three conjugate diametral planes, the plane through the feet of the perpendiculars meets the normal at P at a fixed point whose distance from P is

$$\frac{a^2b^2c^2}{p(b^2c^2+c^2a^2+a^2b^2)},$$

where p is the perpendicular from the centre to the tangent plane at P.

13. Find the locus of centres of sections of a conicoid that are at a constant distance from the centre.

14. Shew that the equations to the right circular cones that pass through the axes (which are rectangular) are $yz \pm zx \pm xy=0$.

Deduce that the lines through a given point P, which are perpendicular to their polars with respect to $x^2/a^2+y^2/b^2+z^2/c^2=1$, lie upon a right circular cone if P lies on one of the lines

$$(b^2-c^2)^2x^2=(c^2-a^2)^2y^2=(a^2-b^2)^2z^2.$$

15. Chords of a conicoid which are parallel to a given diameter and are such that the normals at their extremities intersect, lie in a fixed plane through the given diameter.

16. The perpendiculars from the origin to the faces of the tetrahedron whose vertices are the feet of the four normals from a point to the cone $ax^2+by^2+cz^2=0$, lie on the cone

$$a(b-c)^2x^2+b(c-a)^2y^2+c(a-b)^2z^2=0.$$

17. P, Q, R ; P′, Q′, R′ are the feet of the six normals from a point to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$. Prove that the poles of the planes PQR, P′Q′R′ lie on the surface $\Sigma\left\{\frac{b^2z^2+c^2y^2}{b^2-c^2}(x^2-a^2)\right\}=0$.

18. The normals at P and P′, points of the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1,$$

meet the plane XOY in A and A′ and make angles θ, θ' with PP′. Prove that $\mathrm{PA}\cos\theta+\mathrm{P'A'}\cos\theta'=0$.

19. The normals to $x^2/a^2+y^2/b^2+z^2/c^2=1$ at all points of its intersection with $lyz+mzx+nxy=0$ intersect the line

$$\frac{a^2x}{l(a^2-b^2)(c^2-a^2)}=\frac{b^2y}{m(b^2-c^2)(a^2-b^2)}=\frac{c^2z}{n(c^2-a^2)(b^2-c^2)}.$$

20. Shew that the points on an ellipsoid the normals at which intersect a given straight line lie on the curve of intersection of the ellipsoid and a conicoid.

21. The normals to $x^2/a^2+y^2/b^2+z^2/c^2=1$ at points of its intersection with $x/a+y/b+z/c=1$ lie on the surface

$$\Sigma\left(\frac{ab\,xy+bc\,yz+ca\,zx}{b(a^2-c^2)y+c(a^2-b^2)z}\right)=1.$$

22. Prove that two normals to $ax^2+by^2+cz^2=1$ lie in the plane $lx+my+nz=p$, and that they are at right angles if

$$abcp^2\Sigma\{a(b+c)l^2\}=\Sigma\{a^2(b-c)^2m^2n^2\}.$$

23. The locus of a point, the sum of the squares of whose normal distances from the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ is constant, $(=k^2)$, is

$$6x^2+6y^2+6z^2-2\Sigma\left\{\frac{a^2x^2(b^2+c^2-2a^2)}{(a^2-b^2)(c^2-a^2)}\right\}+2a^2+2b^2+2c^2=k^2.$$

24. If the feet of the six normals from (α, β, γ) are

$$(x_r, y_r, z_r), \quad (r=1, 2, \dots 6),$$

prove that $$a^2\alpha\Sigma\left(\frac{1}{x_r}\right)+b^2\beta\Sigma\left(\frac{1}{y_r}\right)+c^2\gamma\Sigma\left(\frac{1}{z_r}\right)=0.$$

25. If the feet of three of the normals from P to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ lie in the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, the feet of the other three lie in the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+1=0$, and P lies on the line

$$a(b^2-c^2)x=b(c^2-a^2)y=c(a^2-b^2)z.$$

26. If A, B are $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, the pair of tangent planes at the points where AB cuts the conicoid $S \equiv ax^2+by^2+cz^2-1=0$ is given by

$$S_2P_1^2-2P_1P_2P_{12}+S_1P_2^2=0,$$

and the pair of tangent planes that intersect in AB, by

$$S(S_1S_2-P^2_{12})-S_2P_1^2+2P_1P_2P_{12}-S_1P_2^2=0,$$

where

$$S_1 \equiv a\alpha_1^2+b\beta_1^2+c\gamma_1^2-1, \text{ etc.};$$

$$P_1 \equiv a\alpha_1x+b\beta_1y+c\gamma_1z-1, \text{ etc.};$$

$$P_{12} \equiv a\alpha_1\alpha_2+b\beta_1\beta_2+c\gamma_1\gamma_2-1.$$

27. If P, (x_1, y_1, z_1), Q, (x_2, y_2, z_2), R, (x_3, y_3, z_3) are the extremities of three conjugate semi-diameters of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, and $OP=r_1$, $OQ=r_2$, $OR=r_3$, prove that the equation to the sphere OPQR can be written

$$x^2+y^2+z^2-r_1^2\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}\right)-r_2^2\left(\frac{xx_2}{a^2}+\frac{yy_2}{b^2}+\frac{zz_2}{c^2}\right)$$
$$-r_3^2\left(\frac{xx_3}{a^2}+\frac{yy_3}{b^2}+\frac{zz_3}{c^2}\right)=0,$$

and prove that the locus of the centres of spheres through the origin and the extremities of three equal conjugate semi-diameters is

$$12(a^2x^2+b^2y^2+c^2z^2)=(a^2+b^2+c^2)^2.$$

CHAPTER VIII.

THE AXES OF A PLANE SECTION OF A CONICOID.

85. We have proved, (§ 54, Exs. 2, 3), that every plane section of a conicoid is a conic, and that parallel plane sections are similar and similarly situated conics. We now proceed to find equations to determine the magnitudes and directions of the axes of a given plane section of a given conicoid.

General method for determining the axes. If the lengths of the axes of a conic are 2α and 2β, and $\alpha > r > \beta$, the conic has two diameters of length $2r$, and they are equally inclined to the axes. If $r = \alpha$ or β, the two diameters of length $2r$ coincide with an axis. Hence to find the axes of the conic in which a given plane cuts a conicoid, we first form the equation to a cone whose vertex is the centre, C, of the conic and which has as generators the lines of length r which can be drawn in the plane from C to the conicoid. The lines of section of this cone and the given plane are the semi-diameters of length r of the conic. If $2r$ is the length of an axis, these are coincident, or the plane touches the cone, the generator of contact being the axis. The condition of tangency gives an equation which determines r: the comparison of the equations of the given plane and a tangent plane to the cone leads to the direction-cosines of the generator of contact.

86. Axes of the section of a central conicoid by a plane through the centre. Let the equations, referred to rectangular axes, of the conicoid and plane be

$$ax^2 + by^2 + cz^2 = 1, \quad lx + my + nz = 0.$$

The centre of the conicoid is also the centre of the section. If λ, μ, ν are the direction-cosines of a semi-diameter of the conicoid of length r, the point $(\lambda r, \mu r, \nu r)$ is on the conicoid. Therefore

$$a\lambda^2+b\mu^2+c\nu^2=\frac{1}{r^2}=\frac{\lambda^2+\mu^2+\nu^2}{r^2}.$$

Hence the semi-diameters of the conicoid of length r are generators of the cone

$$x^2(a-1/r^2)+y^2(b-1/r^2)+z^2(c-1/r^2)=0. \qquad \text{.........(1)}$$

The lines of section of the cone and plane are the semi-diameters of the conic of length r. Hence, if r is the length of either semi-axis of the conic in which the plane $lx+my+nz=0$ cuts the conicoid, the plane touches the cone, and therefore

$$\frac{l^2}{ar^2-1}+\frac{m^2}{br^2-1}+\frac{n^2}{cr^2-1}=0, \qquad \text{................(2)}$$

or $r^4(bcl^2+cam^2+abn^2)-r^2\{(b+c)l^2+(c+a)m^2+(a+b)n^2\}$
$$+(l^2+m^2+n^2)=0.$$

The roots of this quadratic in r^2 give the squares of the semi-axes of the section.

If $2r$ is the length of an axis and λ, μ, ν are the direction-cosines, the given plane touches the cone (1) along the line $x/\lambda=y/\mu=z/\nu$, and therefore is represented by the equation

$$\lambda x(a-1/r^2)+\mu y(b-1/r^2)+\nu z(c-1/r^2)=0.$$

Therefore $$\frac{\lambda(ar^2-1)}{l}=\frac{\mu(br^2-1)}{m}=\frac{\nu(cr^2-1)}{n}. \qquad \text{...........(3)}$$

These determine the direction-cosines of the axis of length $2r$.

Since the extremities of the semi-diameters of length r of the conicoid lie upon the sphere $x^2+y^2+z^2=r^2$, the equation of the cone through them may be obtained by making the equation to the conicoid homogeneous by means of the equation to the sphere. Thus the cone is

$$ax^2+by^2+cz^2=\frac{x^2+y^2+z^2}{r^2},$$

which is another form of equation (1).

Ex. 1. Prove that the axes of the section of the conicoid $ax^2+by^2+cz^2=1$ by the plane $lx+my+nz=0$ lie on the cone

$$(b-c)\frac{l}{x}+(c-a)\frac{m}{y}+(a-b)\frac{n}{z}=0.$$

(From equations (3) we deduce that

$$(b-c)\frac{l}{\lambda}+(c-a)\frac{m}{\mu}+(a-b)\frac{n}{\nu}=0.)$$

Ex. 2. Prove that the cone of Ex. 1 passes through the normal to the plane of section and the diameter to which the plane of section is diametral plane. Prove also that the cone passes through two sets of conjugate diameters of the conicoid. (Cf. Ex. 6, § 77.)

Ex. 3. Find the lengths of the axes of the conics given by

$$\text{(i)}\quad 3x^2+2y^2+6z^2=1, \qquad x+y+z=0;$$
$$\text{(ii)}\quad 2x^2+y^2-z^2=1, \quad 3x+4y+5z=0.$$

Ans. (i) ·64, ·45; (ii) 3·08, ·76.

Ex. 4. Prove that the equation of the conic

$$x^2+2y^2-2z^2=1, \quad 3x-2y-z=0,$$

referred to its principal axes, is approximately

$$1{\cdot}70x^2-1{\cdot}77y^2=1.$$

Ex. 5. Prove that the lengths of the axes of the section of the ellipsoid of revolution $\dfrac{x^2+y^2}{a^2}+\dfrac{z^2}{c^2}=1$, by the plane $lx+my+nz=0$, are

$$a, \quad ac(l^2+m^2+n^2)^{\frac{1}{2}}\{a^2(l^2+m^2)+c^2n^2\}^{-\frac{1}{2}},$$

and that the equations to the axes are

$$\frac{x}{m}=\frac{y}{-l}=\frac{z}{0}; \quad \frac{x}{nl}=\frac{y}{mn}=\frac{z}{-(l^2+m^2)}.$$

Ex. 6. Prove that the area of the section of the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1$$

by the plane $lx+my+nz=0$ is $\dfrac{\pi abc}{p}$, where p is the perpendicular from the centre to the tangent plane which is parallel to the given plane.

Ex. 7. The section of the conicoid $ax^2+by^2+cz^2=1$ by a tangent plane to the cone

$$\frac{x^2}{b+c}+\frac{y^2}{c+a}+\frac{z^2}{a+b}=0$$

is a rectangular hyperbola.

Ex. 8. The section of a hyperboloid of one sheet by a tangent plane to the asymptotic cone is two parallel straight lines. What is the corresponding section of the hyperboloid of two sheets which has the same asymptotic cone?

Ex. 9. Central sections of an ellipsoid whose area is constant envelope a cone of the second degree.

Ex. 10. If A_1, A_2, A_3 are the areas of three mutually perpendicular central sections of an ellipsoid, $A_1^{-2}+A_2^{-2}+A_3^{-2}$ is constant.

Ex. 11. One of the axes of each section of the ellipsoid $\dfrac{x^2+y^2}{a^2}+\dfrac{z^2}{c^2}=1$ by a tangent plane to the cone $y^2+z^2=x^2$ lies on the cone

$$(x^2+y^2)^2=z^2(x^2-y^2).$$

What is the nature of the section of this cone by a plane parallel to the plane XOY? Sketch the form of the cone.

Ex. 12. Prove that the axes of sections of the conicoid

$$ax^2+by^2+cz^2=1$$

which pass through the line $\dfrac{x}{l}=\dfrac{y}{m}=\dfrac{z}{n}$ lie on the cone

$$\frac{b-c}{x}(mz-ny)+\frac{c-a}{y}(nx-lz)+\frac{a-b}{z}(ly-mx)=0.$$

87. Axes of any section of a central conicoid. Let the equations, referred to rectangular axes, of the conicoid and plane be $\quad ax^2+by^2+cz^2=1, \quad lx+my+nz=p.$

Then if C, (α, β, γ) is the centre of the section, the plane is also represented by the equation

$$(x-\alpha)a\alpha+(y-\beta)b\beta+(z-\gamma)c\gamma=0.$$

Therefore $$\frac{a\alpha}{l}=\frac{b\beta}{m}=\frac{c\gamma}{n}=\frac{a\alpha^2+b\beta^2+c\gamma^2}{p}. \quad \text{.................(1)}$$

Hence $a\alpha^2+b\beta^2+c\gamma^2=\dfrac{p^2}{l^2/a+m^2/b+n^2/c}=\dfrac{p^2}{p_0^{\,2}}$, say.

The equation to the conicoid referred to parallel axes through C is

$$ax^2+by^2+cz^2+2(a\alpha x+b\beta y+c\gamma z)+a\alpha^2+b\beta^2+c\gamma^2-1=0,$$

or $$ax^2+by^2+cz^2+2(a\alpha x+b\beta y+c\gamma z)-k^2=0,$$

where $$k^2\equiv 1-\frac{p^2}{p_0^{\,2}}.$$

The equation to the plane is now $lx+my+nz=0$.

If λ, μ, ν are the direction-cosines of a line of length r drawn from C to the conicoid,

$$r^2(a\lambda^2+b\mu^2+c\nu^2)+2r(a\alpha\lambda+b\beta\mu+c\gamma\nu)-k^2=0.$$

If the line lies in the given plane

$$l\lambda+m\mu+n\nu=0,$$

and therefore, by (1), $a\alpha\lambda+b\beta\mu+c\gamma\nu=0.$

Hence $\quad r^2(a\lambda^2+b\mu^2+c\nu^2)-k^2(\lambda^2+\mu^2+\nu^2)=0,$

and therefore the semi-diameters of the section of length r lie on the cone

$$x^2\left(\frac{ar^2}{k^2}-1\right)+y^2\left(\frac{br^2}{k^2}-1\right)+z^2\left(\frac{cr^2}{k^2}-1\right)=0.$$

If r is the length of either semi-axis of the section, the plane touches the cone. Therefore

$$\frac{l^2}{\dfrac{ar^2}{k^2}-1}+\frac{m^2}{\dfrac{br^2}{k^2}-1}+\frac{n^2}{\dfrac{cr^2}{k^2}-1}=0. \quad \ldots\ldots\ldots\ldots\ldots(2)$$

And, as in § 86, the direction-cosines of the axis of length $2r$ are given by

$$\frac{\lambda\left(\dfrac{ar^2}{k^2}-1\right)}{l}=\frac{\mu\left(\dfrac{br^2}{k^2}-1\right)}{m}=\frac{\nu\left(\dfrac{cr^2}{k^2}-1\right)}{n}. \quad \ldots\ldots\ldots(3)$$

Comparing these equations with equations (2) and (3) of § 86, we see that if α and β are the lengths of the semi-axes of the section by the plane $lx+my+nz=0$, the semi-axes of the section by the plane $lx+my+nz=p$ are $k\alpha$ and $k\beta$, or $\quad \alpha\sqrt{1-\dfrac{p^2}{p_0^{\,2}}},\ \beta\sqrt{1-\dfrac{p^2}{p_0^{\,2}}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$

and that the corresponding axes are parallel. We thus have another proof for central surfaces of the proposition that parallel plane sections are similar and similarly situated conics.

From equations (4) it follows that if A, A_0 are the areas of the sections of a conicoid by a given plane and the parallel plane through the centre, $\mathsf{A}=\mathsf{A}_0\left(1-\dfrac{p^2}{p_0^{\,2}}\right)$, where p, p_0 are the perpendiculars from the centre to the given plane and the parallel tangent plane. Thus the area of the section of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ by the plane

$$lx+my+nz=p$$

is $\quad \dfrac{\pi abc(l^2+m^2+n^2)^{\frac{1}{2}}}{(a^2l^2+b^2m^2+c^2n^2)^{\frac{1}{2}}}\left\{1-\dfrac{p^2}{a^2l^2+b^2m^2+c^2n^2}\right\}.$

The student should note that the equation to the cone through *all* the lines of length r drawn from C to the conicoid would be obtained by making the equation

$$ax^2+by^2+cz^2+2(a\alpha x+b\beta y+c\gamma z)-k^2=0$$

homogeneous by means of the equation $x^2+y^2+z^2=r^2$. It would be of the *fourth* degree, while for our purpose we require a cone of the second degree. The cone chosen passes through the lines of length r which lie in the given plane, and these lines alone need be considered.

Ex. 1. OP is a given semi-diameter of a conicoid and OA$(=\alpha)$, OB$(=\beta)$, are the principal semi-axes of the section of the diametral plane of OP. A plane parallel to AOB meets OP in C. Prove that the principal axes of the section of the conicoid by this plane are $\alpha\sqrt{1-\text{OC}^2/\text{OP}^2}$, $\beta\sqrt{1-\text{OC}^2/\text{OP}^2}$, and deduce equations (4), § 87.

(Take OP, OA, OB, as coordinate axes.)

Ex. 2. Find the coordinates of the centre and the lengths of the axes of the section of the ellipsoid $3x^2+3y^2+6z^2=10$ by the plane $x+y+z=1$.

Ans. $\left(\frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right)$; $\sqrt{\frac{44}{15}}, \frac{\sqrt{44}}{5}$.

Ex. 3. If OP, OQ, OR are conjugate semi-diameters of an ellipsoid prove that the area of the section of the ellipsoid by the plane PQR is two-thirds the area of the parallel central section.

Ex. 4. Find the area of the section of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ by the plane $x/a+y/b+z/c=1$.

Ans. $\frac{2\pi}{3\sqrt{3}}(b^2c^2+c^2a^2+a^2b^2)^{\frac{1}{2}}$.

Ex. 5. Find the locus of the centres of sections of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ whose area is constant, $(=\pi k^2)$.

Ans. $a^2b^2c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right)^2=k^4\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)$.

Ex. 6. Prove that tangent planes to $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}+1=0$ which cut $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}-1=0$ in ellipses of constant area πk^2 have their points of contact on the surface $\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}=\frac{k^4}{4a^2b^2c^2}$.

Ex. 7. Prove that the axes of the section whose centre is P are the straight lines in which the plane of section cuts the cone containing the normals from P.

Ex. 8. Find the lengths of the axes of the sections of the surface $4yz+5zx-5xy=8$ by the planes (i) $x+y-z=0$, (ii) $2x+y-z=0$.

Ans. (i) 2, $\sqrt{3}$ · (ii) 2, 2.

Ex. 9. Prove that the axes of the section of

$$f(x, y, z)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1$$

by the plane $lx+my+nz=0$ are given by

$$r^4(\mathsf{A}l^2 \ldots + 2\mathsf{F}mn \ldots) + r^2\{f(l, m, n) - (a+b+c)(l^2+m^2+n^2)\} + (l^2+m^2+n^2) = 0,$$

where $\mathsf{A} \equiv bc - f^2$, etc.

Prove also that the axes are the lines in which the plane cuts the cone

$$(mz-ny)\frac{\partial f}{\partial x} + (nx-lz)\frac{\partial f}{\partial y} + (ly-mx)\frac{\partial f}{\partial z} = 0.$$

Ex. 10. Prove that the axes of the section of the cone

$$ax^2+by^2+cz^2=0$$

by the plane $lx+my+nz=p$ are given by

$$\frac{l^2}{ap_0^2r^2+p^2} + \frac{m^2}{bp_0^2r^2+p^2} + \frac{n^2}{cp_0^2r^2+p^2} = 0,$$

where $p_0^2 \equiv \dfrac{l^2}{a} + \dfrac{m^2}{b} + \dfrac{n^2}{c}$.

88. Axes of a given section of a paraboloid. If the equations to the plane and the paraboloid are

$$lx+my+nz=p, \quad ax^2+by^2=2z,$$

the centre of the section, (α, β, γ), is given by

$$\frac{a\alpha}{l} = \frac{b\beta}{m} = \frac{-1}{n} = \frac{a\alpha^2+b\beta^2-\gamma}{p}.$$

Whence $\quad a\alpha^2+b\beta^2-2\gamma = \dfrac{l^2/a+m^2/b+2np}{-n^2} = \dfrac{p_0^2}{-n^2}$, say.

Changing the origin to (α, β, γ) and proceeding as in § 87, we find that λ, μ, ν, the direction-cosines of a semi-diameter of length r of the section, satisfy the equation

$$n^2r^2(a\lambda^2+b\mu^2) - p_0^2(\lambda^2+\mu^2+\nu^2) = 0.$$

The semi-diameters are therefore the lines in which the plane cuts the cone

$$x^2(an^2r^2-p_0^2) + y^2(bn^2r^2-p_0^2) - z^2p_0^2 = 0.$$

Hence the lengths of the axes are given by

$$\frac{l^2}{an^2r^2-p_0^2} + \frac{m^2}{bn^2r^2-p_0^2} - \frac{n^2}{p_0^2} = 0,$$

or

$$abn^6r^4 - n^2r^2p_0^2\{(a+b)n^2+am^2+bl^2\} + p_0^4(l^2+m^2+n^2) = 0,$$

and the direction cosines by

$$\frac{\lambda(an^2r^2-p_0^2)}{l} = \frac{\mu(bn^2r^2-p_0^2)}{m} = \frac{\nu p_0^2}{-n}.$$

Ex. 1. Find the lengths of the axes of the section of the paraboloid $2x^2+y^2=z$ by the plane $x+2y+z=4$. *Ans.* 5·28, 1·68.

Ex. 2. A plane section through the vertex of the paraboloid of revolution $x^2+y^2=2az$ makes an angle θ with the axis of the surface. Prove that its principal axes are $a\cot\theta\operatorname{cosec}\theta$, $a\cot\theta$.

Ex. 3. Prove that the axes of the section of the paraboloid $xy=az$ by the plane $lx+my+nz=0$ are given by

$$n^6r^4-4a^2r^2l^2m^2n^2-4a^4l^2m^2(l^2+m^2+n^2)=0.$$

Ex. 4. Find the locus of the centres of sections of the paraboloid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=2z$ which are of constant area πk^2.

Ans. $a^2b^2\left(\dfrac{x^2}{a^4}+\dfrac{y^2}{b^4}+1\right)\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-2z\right)^2=k^4.$

Ex. 5. Given that the radius of curvature at a point P of a conic whose centre is C is equal to $CD^3/\alpha\beta$, where α and β are the axes and CD is the semi-diameter conjugate to CP, find the radius of curvature at the origin of the conic $ax^2+by^2=2z$, $lx+my+nz=0$.

Ans. $(l^2+m^2)^{\frac{3}{2}}(am^2+bl^2)^{-1}(l^2+m^2+n^2)^{-\frac{1}{2}}.$

Ex. 6. Planes are drawn through a fixed point (α, β, γ) so that their sections of the paraboloid $ax^2+by^2=2z$ are rectangular hyperbolas. Prove that they touch the cone

$$\frac{(x-\alpha)^2}{b}+\frac{(y-\beta)^2}{a}+\frac{(z-\gamma)^2}{a+b}=0.$$

CIRCULAR SECTIONS.

89. If $F=0$, the equation to a conicoid, can be thrown into the form $S+\lambda uv=0$, where $S=0$ is the equation to a sphere and $u=0$, $v=0$ represent planes, the common points of the conicoid and planes lie on the sphere, and therefore the sections of the conicoid by the planes are circles.

90. The circular sections of an ellipsoid. The equation, referred to rectangular axes, of the ellipsoid,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

can be written in the forms

$$\frac{x^2+y^2+z^2}{a^2}-1+y^2\left(\frac{1}{b^2}-\frac{1}{a^2}\right)+z^2\left(\frac{1}{c^2}-\frac{1}{a^2}\right)=0,$$

$$\frac{x^2+y^2+z^2}{b^2}-1+z^2\left(\frac{1}{c^2}-\frac{1}{b^2}\right)+x^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)=0,$$

$$\frac{x^2+y^2+z^2}{c^2}-1+x^2\left(\frac{1}{a^2}-\frac{1}{c^2}\right)+y^2\left(\frac{1}{b^2}-\frac{1}{c^2}\right)=0.$$

Hence the planes

$$\text{(i)}\quad y^2\left(\frac{1}{b^2}-\frac{1}{a^2}\right)+z^2\left(\frac{1}{c^2}-\frac{1}{a^2}\right)=0,$$

$$\text{(ii)}\quad z^2\left(\frac{1}{c^2}-\frac{1}{b^2}\right)+x^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)=0,$$

$$\text{(iii)}\quad x^2\left(\frac{1}{a^2}-\frac{1}{c^2}\right)+y^2\left(\frac{1}{b^2}-\frac{1}{c^2}\right)=0$$

cut the ellipsoid in circles of radii a, b, c respectively. If $a>b>c$, only the second of these equations gives real planes, and therefore the only real central circular sections of the ellipsoid pass through the mean axis, and are given by the equations

$$\frac{x}{a}\sqrt{a^2-b^2}\pm\frac{z}{c}\sqrt{b^2-c^2}=0.$$

Since parallel plane sections are similar and similarly situated conics, the equations

$$\frac{x}{a}\sqrt{a^2-b^2}+\frac{z}{c}\sqrt{b^2-c^2}=\lambda,\quad \frac{x}{a}\sqrt{a^2-b^2}-\frac{z}{c}\sqrt{b^2-c^2}=\mu$$

give circular sections for all values of λ and μ.

91. *Any two circular sections of an ellipsoid which are not parallel lie on a sphere.*

The equation $\quad k\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}-1\right)+$

$$\left(\frac{x}{a}\sqrt{a^2-b^2}+\frac{z}{c}\sqrt{b^2-c^2}-\lambda\right)\left(\frac{x}{a}\sqrt{a^2-b^2}-\frac{z}{c}\sqrt{b^2-c^2}-\mu\right)=0$$

represents a conicoid which passes through the sections, and if $k=b^2$, the equation becomes

$$x^2+y^2+z^2-\frac{(\lambda+\mu)\sqrt{a^2-b^2}}{a}x+\frac{(\lambda-\mu)\sqrt{b^2-c^2}}{c}z+\lambda\mu-b^2=0,$$

which represents a sphere.

92. Circular sections of the hyperboloids. By the method of § 90, we deduce that the real central circular sections of the hyperboloids

$$\text{(i)}\ \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,\quad \text{(ii)}\ \frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,\quad (a>b>c)$$

are given by

$$\text{(i) } \frac{y}{b}\sqrt{a^2-b^2} \pm \frac{z}{c}\sqrt{a^2+c^2}=0, \quad \text{(ii) } \frac{x}{a}\sqrt{a^2+b^2} \pm \frac{z}{c}\sqrt{b^2-c^2}=0.$$

The radius of the central circular sections of the hyperboloid of one sheet is a. The planes given by

$$\frac{x}{a}\sqrt{a^2+b^2} \pm \frac{z}{c}\sqrt{b^2-c^2}=0$$

do not meet the hyperboloid of two sheets in any real points. They are the planes through the centre parallel to systems of planes which cut the surface in real circles.

Ex. 1. Prove that the section of the hyperboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ by the plane $\frac{x}{a}\sqrt{a^2+b^2}+\frac{z}{c}\sqrt{b^2-c^2}=\lambda$ is real if $\lambda^2 > a^2+c^2$.

Ex. 2. Find the real central circular sections of the ellipsoid $x^2+2y^2+6z^2=8$. *Ans.* $x^2-4z^2=0$.

Ex. 3. Prove that the planes $2x+3z-5=0$, $2x-3z+7=0$ meet the hyperboloid $-x^2+3y^2+12z^2=75$ in circles which lie on the sphere

$$3x^2+3y^2+3z^2+4x+36z-110=0.$$

Ex. 4. Prove that the radius of the circle in which the plane

$$\frac{x}{a}\sqrt{a^2-b^2}+\frac{z}{c}\sqrt{b^2-c^2}=\lambda$$

cuts the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is $b\sqrt{1-\frac{\lambda^2}{a^2-c^2}}$.

Ex. 5. Find the locus of the centres of spheres of constant radius k which cut the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ in a pair of circles. (Use § 91.)

Ans. $y=0$, $\frac{x^2}{a^2-b^2}-\frac{z^2}{b^2-c^2}=1-\frac{k^2}{b^2}$.

Ex. 6. Chords of the ellipse $x^2/a^2+y^2/b^2=1$, $z=0$, are drawn so as to make equal angles with its axes, and on them as diameters circles are described whose planes are parallel to OZ. Prove that these circles generate the ellipsoid $2b^2x^2+2a^2y^2+(a^2+b^2)z^2=2a^2b^2$.

93. Circular sections of any central conicoid. An equation of the form

$$f(x, y, z) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1$$

represents a central conicoid. It may be written

$$f(x, y, z)-\lambda(x^2+y^2+z^2)+\lambda\left(x^2+y^2+z^2-\frac{1}{\lambda}\right)=0.$$

Hence if $\quad f(x, y, z)-\lambda(x^2+y^2+z^2)=0$

represents a pair of planes, these planes cut the conicoid in circles. For a pair of planes

$$\begin{vmatrix} a-\lambda, & h, & g \\ h, & b-\lambda, & f \\ g, & f, & c-\lambda \end{vmatrix}=0.$$

This equation gives three values of λ. It will be proved later, § 145, that these are always real, and that only the mean value gives real planes.

Ex. 1. Find the real central circular sections of the conicoid

$$3x^2+5y^2+3z^2+2xz=4.$$

The equation may be written

$$3x^2+5y^2+3z^2+2xz-\lambda(x^2+y^2+z^2)+\lambda(x^2+y^2+z^2)-4=0.$$

If $3x^2+5y^2+3z^2+2xz-\lambda(x^2+y^2+z^2)=0$ represents a pair of planes, $\lambda^3-11\lambda^2+38\lambda-40=0$, or $\lambda=2$, 4, or 5. For these values of λ the equation to the planes becomes

$$(x+z)^2+3y^2=0, \quad (x-z)^2-y^2=0, \quad x^2-xz+z^2=0.$$

The real circular sections correspond therefore to $\lambda=4$ and have equations

$$x-z+y=0, \quad x-z-y=0.$$

Ex. 2. Find the equations to the real central circular sections of the conicoids,

(i) $5y^2-8z^2+18yz-14zx-10xy+27=0$,

(ii) $2x^2+5y^2+2z^2-yz-4zx-xy+4=0$,

(iii) $6x^2+13y^2+6z^2-10yz+4zx-10xy=1$.

Ans. (i) $(x-2y-5z)(3x-4y+z)=0$,

(ii) $(x+y+z)(2x-y+2z)=0$,

(iii) $2(x+z)^2-10y(x+z)+9y^2=0$.

Ex. 3. Find the equations to the circular sections of the conicoid

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)+1=0.$$

Ans. $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=\lambda, \quad ax+by+cz=\mu.$

Ex. 4. Find the conditions that the equations

$$f(x, y, z)=1, \quad lx+my+nz=0$$

should determine a circle.

The equation $f(x, y, z)-\lambda(x^2+y^2+z^2)=0$ is to represent two planes, one of which is the given plane. Therefore

$$f(x, y, z)-\lambda(x^2+y^2+z^2)\equiv(lx+my+nz)\left\{(a-\lambda)\frac{x}{l}+(b-\lambda)\frac{y}{m}+(c-\lambda)\frac{z}{n}\right\}.$$

Whence, comparing coefficients of yz, zx, xy, we obtain

$$\lambda=\frac{bn^2+cm^2-2fmn}{m^2+n^2}=\frac{cl^2+an^2-2gnl}{n^2+l^2}=\frac{am^2+bl^2-2hlm}{l^2+m^2}.$$

(We assume here that l, m, n are all different from zero. If $l=0$, the conditions become $(\lambda=a)$, $g=h=0$, $(c-a)m^2-2fmn+(b-a)n^2=0$.)

94. Circular sections of the paraboloids. The equation $ax^2+by^2=2z$ may be written in the forms,

$$a\left(x^2+y^2+z^2-\frac{2z}{a}\right)-y^2(a-b)-az^2=0,$$

$$b\left(x^2+y^2+z^2-\frac{2z}{b}\right)-x^2(b-a)-bz^2=0,$$

$$ax^2+by^2-(0\cdot x^2+0\cdot y^2+0\cdot z^2+2z)=0.$$

Hence if $a>b>0$, $x^2(a-b)=bz^2$ represents real planes which meet the paraboloid in circles, and the systems of circular sections are given by

$$x\sqrt{a-b}+z\sqrt{b}=\lambda, \quad x\sqrt{a-b}-z\sqrt{b}=\mu.$$

If, however, a or b is negative, the only real planes are those given by $ax^2+by^2=0$. The equation

$$0\cdot x^2+0\cdot y^2+0\cdot z^2+2z=0$$

is the limiting form of

$$kx^2+ky^2+k\left(z+\frac{1}{k}\right)^2=\frac{1}{k}$$

as k tends to zero, and therefore the sphere containing the circular sections is in this case of infinite radius, and the circular sections are circles of infinite radius, *i.e.* straight lines. They are the straight lines in which the plane $z=0$ cuts the surface, (§ 79).

Ex. 1. Find the circular sections of the paraboloid $x^2+10z^2=2y$.
Ans. $y\pm 3z=\lambda$.

Ex. 2. Find the radius of the circle in which the plane $7x+2z=5$ cuts the paraboloid $53x^2+4y^2=8z$. *Ans.* $r^2=\frac{314}{53}$.

95. Umbilics. The centres of a series of parallel plane sections of a conicoid lie upon a diameter of the conicoid and the tangent plane at an extremity of the diameter is parallel to the plane sections. If, therefore, P and P′ (fig. 40) are the extremities of the diameter which passes through the centres of a system of circular sections of an ellipsoid, the tangent planes at P and P′ are the limiting positions of the cutting planes, and P and P′ may be regarded as circular sections of zero radius. A circular section of zero

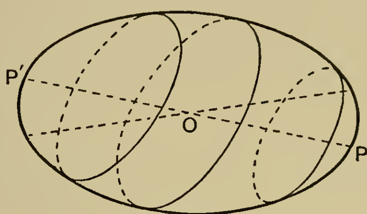

FIG. 40.

radius is called an **umbilic**. It is evident from the form of the hyperboloid of one sheet that the smallest closed section is the principal elliptic section and that the surface has therefore no real umbilics.

To find the umbilics of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

If P, (ξ, η, ζ) is an umbilic, the diametral plane of OP is a central circular section. Therefore the equations

$$\frac{x\xi}{a^2}+\frac{y\eta}{b^2}+\frac{z\zeta}{c^2}=0, \quad \frac{x}{a}\sqrt{a^2-b^2}\pm\frac{z}{c}\sqrt{b^2-c^2}=0$$

represent the same plane. Hence

$$\frac{\xi/a}{\sqrt{a^2-b^2}}=\frac{\eta/b}{0}=\frac{\zeta/c}{\pm\sqrt{b^2-c^2}}$$

$$=\frac{1}{\pm\sqrt{a^2-c^2}}, \quad \text{since} \quad \frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2}=1,$$

and therefore

$$\xi=\frac{\pm a\sqrt{a^2-b^2}}{\sqrt{a^2-c^2}}, \quad \eta=0, \quad \zeta=\frac{\pm c\sqrt{b^2-c^2}}{\sqrt{a^2-c^2}}.$$

These give the coordinates of the four umbilics.

The umbilics of the hyperboloid of two sheets

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

are real and given by

$$\xi=\frac{\pm a\sqrt{a^2+b^2}}{\sqrt{a^2+c^2}}, \quad \eta=0, \quad \zeta=\frac{\pm c\sqrt{b^2-c^2}}{\sqrt{a^2+c^2}}.$$

Ex. 1. Prove that the umbilics of the ellipsoid lie on the sphere

$$x^2+y^2+z^2=a^2-b^2+c^2.$$

Ex. 2. Prove that the perpendicular distance from the centre to the tangent plane at an umbilic of the ellipsoid is ac/b.

Ex. 3. Prove that the central circular sections of the conicoid $(a-b)x^2+ay^2+(a+b)z^2=1$ are at right angles and that the umbilics are given by $x=\pm\sqrt{\dfrac{a+b}{2a(a-b)}}$, $y=0$, $z=\pm\sqrt{\dfrac{a-b}{2a(a+b)}}$.

Ex. 4. Prove that the umbilics of the conicoid $\dfrac{x^2}{a+b}+\dfrac{y^2}{a}+\dfrac{z^2}{a-b}=1$ are the extremities of the equal conjugate diameters of the ellipse

$$y=0, \quad \frac{x^2}{a+b}+\frac{z^2}{a-b}=1.$$

Ex. 5. Prove that the umbilics of the paraboloid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=2z$, $a>b$, are $\left(0, \ \pm b\sqrt{a^2-b^2}, \ \dfrac{a^2-b^2}{2}\right)$.

Ex. 6. Deduce the coordinates of the umbilics of the ellipsoid from the result of Ex. 4, § 92.

*Examples V.

1. Prove that if λ_1, μ_1, ν_1 ; λ_2, μ_2, ν_2 are the direction-cosines of the axes of any plane section of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$,

$$\frac{\lambda_1\lambda_2}{a^2(b^2-c^2)}=\frac{\mu_1\mu_2}{b^2(c^2-a^2)}=\frac{\nu_1\nu_2}{c^2(a^2-b^2)}.$$

2. If $\Delta_1, \Delta_2, \Delta_3$; $\delta_1, \delta_2, \delta_3$ are the areas of the sections of the ellipsoids $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$, $\dfrac{x^2}{\alpha^2}+\dfrac{y^2}{\beta^2}+\dfrac{z^2}{\gamma^2}=1$, by three conjugate diametral planes of the former,

$$\frac{\Delta_1^2}{\delta_1^2}+\frac{\Delta_2^2}{\delta_2^2}+\frac{\Delta_3^2}{\delta_3^2}=\frac{a^2b^2c^2}{\alpha^2\beta^2\gamma^2}\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right).$$

3. If A_1, A_2, A_3 are the areas of the sections of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

by the diametral planes of three mutually perpendicular semi-diameters of lengths r_1, r_2, r_3,

$$\frac{A_1^2}{r_1^2}+\frac{A_2^2}{r_2^2}+\frac{A_3^2}{r_3^2}=\pi^2\left(\frac{b^2c^2}{a^2}+\frac{c^2a^2}{b^2}+\frac{a^2b^2}{c^2}\right).$$

4. Through a given point (α, β, γ) planes are drawn parallel to three conjugate diametral planes of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Shew that the sum of the ratios of the areas of the sections by these planes to the areas of the parallel diametral planes is $3-\frac{\alpha^2}{a^2}-\frac{\beta^2}{b^2}-\frac{\gamma^2}{c^2}$.

5. Prove that the areas of the sections of greatest and least area of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which pass through the fixed line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ are $\frac{\pi abc}{r_1}$, $\frac{\pi abc}{r_2}$, where r_1, r_2 are the axes of the section by the plane $\frac{lx}{a}+\frac{my}{b}+\frac{nz}{c}=0$.

6. Prove that the systems of circular sections of the cone

$$ax^2+by^2+cz^2=0, \quad a>b>c,$$

are given by $x\sqrt{a-b}\pm z\sqrt{b-c}=\lambda$, and that these also give circular sections of the cone $(a+\mu)x^2+(b+\mu)y^2+(c+\mu)z^2=0$.

7. Any tangent plane to a cone cuts the cyclic planes in lines equally inclined to the generator of contact.

8. Any pair of tangent planes to the cone $ax^2+by^2+cz^2=0$ cuts the cyclic planes $x\sqrt{a-b}\pm z\sqrt{b-c}=0$ in lines which lie upon a right circular cone whose axis is at right angles to the plane of contact.

9. The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ cuts a series of central conicoids whose principal planes are the coordinate planes in rectangular hyperbolas. Shew that the pole of the plane with respect to the conicoids lies on a cone whose section by the given plane is a circle.

10. OP, OQ, OR are conjugate diameters of an ellipsoid, axes a, b, c, and S is the foot of the perpendicular from O to the plane PQR. Shew that the cone whose vertex is S and base is the section of the ellipsoid by the diametral plane parallel to the plane PQR has constant volume $\pi abc/3\sqrt{3}$.

11. If two cones have the same systems of circular sections, their common tangent planes touch them along perpendicular generators.

12. The normals to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ at all points of a central circular section are parallel to a plane that makes an angle $\cos^{-1}\frac{ac}{b\sqrt{a^2-b^2+c^2}}$ with the section.

13. If r_1, r_2 are the axes of a central section of an ellipsoid, and θ_1, θ_2 the angles between the section and the circular sections,

$$\sin\theta_1 \,.\, \sin\theta_2 = \frac{a^2c^2}{a^2-c^2}\left(\frac{1}{r_1^2} \sim \frac{1}{r_2^2}\right),$$

where a and c are the greatest and least axes of the ellipsoid.

14. Through a fixed point which is the pole of a circular section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ are drawn planes cutting the surface in rectangular hyperbolas. Shew that the centres of these hyperbolas lie on a fixed circle whose plane is parallel to one system of circular sections.

15. The locus of the centres of sections of the cone $ax^2+by^2+cz^2=0$, such that the sum of the squares of their axes is constant, $(=k^2)$, is the conicoid

$$a\left(\frac{1}{b}+\frac{1}{c}\right)x^2+b\left(\frac{1}{c}+\frac{1}{a}\right)y^2+c\left(\frac{1}{a}+\frac{1}{b}\right)z^2+k^2=0.$$

16. The area of a central section of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is constant. Shew that the axes of the section lie on the cone

$$\Sigma\frac{a^2-p^2}{a^4}\left(\frac{c^2-a^2}{c^2}\frac{z}{y}-\frac{a^2-b^2}{b^2}\frac{y}{z}\right)^2=0,$$

where p is the distance from the centre of a tangent plane parallel to any of the planes of section.

17. Prove that the tangents at the vertices to the parabolic sections of the conicoid $ax^2+by^2+cz^2=1$ are parallel to generators of the cone

$$\frac{a(b-c)^2}{x^2}+\frac{b(c-a)^2}{y^2}+\frac{c(a-b)^2}{z^2}=0.$$

18. Prove that the normals to central sections of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

which are of given eccentricity e, lie on the cone

$$\begin{aligned}&a^2b^2c^2(e^2-2)^2(x^2+y^2+z^2)(a^2x^2+b^2y^2+c^2z^2)\\&\qquad=(1-e^2)\{a^2(b^2+c^2)x^2+b^2(c^2+a^2)y^2+c^2(a^2+b^2)z^2\}^2\end{aligned}$$

Find the locus of the centres of sections of eccentricity e.

19. Prove that the normal at any point **P** of an ellipsoid is an axis of some plane section of the ellipsoid. If the ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and **P** is the point (x', y', z'), shew that the length of the axis is

$$p^{-3}\left(\frac{x'^2}{a^6}+\frac{y'^2}{b^6}+\frac{z'^2}{c^6}\right)^{-1},$$

where p is the perpendicular from the centre to the tangent plane at **P**.

20. The normal section of an enveloping cylinder of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ has a given area πk^2. Prove that the plane of contact of the cylinder and ellipsoid touches the cone

$$\frac{x^2}{a^4(b^2c^2-k^4)}+\frac{y^2}{b^4(c^2a^2-k^4)}+\frac{z^2}{c^4(a^2b^2-k^4)}=0.$$

21. Prove that the locus of the foci of parabolic sections of the paraboloid $ax^2+by^2=2z$ is

$$ab(2z-ax^2-by^2)(ax^2+by^2)=a^2x^2+b^2y^2.$$

22. Prove that the equation to a conicoid referred to the tangent plane and normal at an umbilic as xy-plane and z-axis is

$$a(x^2+y^2)+cz^2+2fyz+2gzx+2wz=0.$$

If a variable sphere be described to touch a given conicoid at an umbilic, it meets the conicoid in a circle whose plane moves parallel to itself as the radius of the sphere varies.

23. If through the centre of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ a perpendicular is drawn to any central section and lengths equal to the axes of the section are marked off along the perpendicular, the locus of their extremities is given by

$$\frac{a^2x^2}{r^2-a^2}+\frac{b^2y^2}{r^2-b^2}+\frac{c^2z^2}{r^2-c^2}=0,$$

where $r^2\equiv x^2+y^2+z^2$. (The locus is the *Wave Surface.*)

24. Prove that the asymptotes of sections of the conicoid

$$ax^2+by^2+cz^2=1$$

which pass through the line $x=h$, $y=0$ lie on the surface

$$\{ax(x-h)+by^2\}^2+cz^2\{a(x-h)^2+by^2\}=0.$$

25. If the section of the cone whose vertex is P, (α, β, γ) and base $z=0$, $ax^2+by^2=1$, by the plane $x=0$ is a circle, then P lies on the conic $y=0$, $ax^2-bz^2=1$, and the section of the cone by the plane

$$(a-b)\gamma x-2a\alpha z=0$$

is also a circle.

CHAPTER IX.

GENERATING LINES.

96. Ruled surfaces. In cones and cylinders we have examples of surfaces which are generated by a moving straight line. Such surfaces are called **ruled surfaces.** We shall now prove that the hyperboloid of one sheet and the hyperbolic paraboloid are ruled surfaces.

The equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, which represents a hyperboloid of one sheet, may be written,

$$\left(\frac{x}{a}+\frac{z}{c}\right)\left(\frac{x}{a}-\frac{z}{c}\right)=\left(1+\frac{y}{b}\right)\left(1-\frac{y}{b}\right).$$

Whence it appears that the hyperboloid is the locus of the straight lines whose equations are

$$\frac{x}{a}+\frac{z}{c}=\lambda\left(1+\frac{y}{b}\right),\quad \frac{x}{a}-\frac{z}{c}=\frac{1}{\lambda}\left(1-\frac{y}{b}\right); \quad \text{.........(1)}$$

$$\frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right),\quad \frac{x}{a}+\frac{z}{c}=\frac{1}{\mu}\left(1-\frac{y}{b}\right); \quad \text{.........(2)}$$

where λ and μ are variable parameters. It is obviously impossible to assign values to λ and μ so that the equations (1) become identical with the equations (2). Hence the equations give two distinct systems of lines, no member of one coinciding with any member of the other. As λ assumes in turn all real values the line given by the equations (1) moves so as to completely generate the hyperboloid. Similarly, the line given by the equations (2) moves, as μ varies, so as to generate the hyperboloid. The hyperboloid of one sheet is therefore a ruled surface and

can be generated in two ways by the motion of a straight line. (See fig. 41.)

In like manner the equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z,$$

which represents a hyperbolic paraboloid, may be written

$$\left(\frac{x}{a}+\frac{y}{b}\right)\left(\frac{x}{a}-\frac{y}{b}\right)=2z.$$

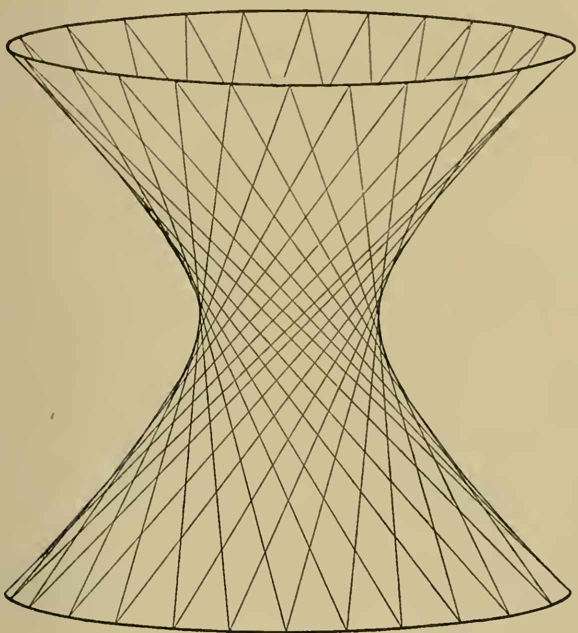

FIG. 41.

Whence it is evident that the paraboloid is the locus of either of the variable lines given by

$$\frac{x}{a}+\frac{y}{b}=\frac{z}{\lambda}, \quad \frac{x}{a}-\frac{y}{b}=2\lambda;$$

$$\frac{x}{a}-\frac{y}{b}=\frac{z}{\mu}, \quad \frac{x}{a}+\frac{y}{b}=2\mu.$$

The hyperbolic paraboloid is therefore a ruled surface

and can be generated in two ways by the motion of a straight line. (See fig. 42.) The generating lines are parallel to one of the fixed planes $\frac{x}{a} \pm \frac{y}{b} = 0$.

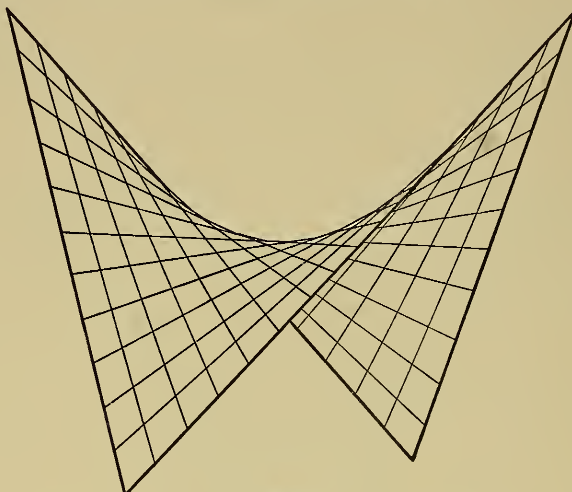

FIG. 42.

Ex. 1. CP, CQ are any conjugate diameters of the ellipse

$$x^2/a^2 + y^2/b^2 = 1, \quad z = c.$$

C′P′, C′Q′ are the conjugate diameters of the ellipse $x^2/a^2 + y^2/b^2 = 1$, $z = -c$, drawn in the same directions as CP and CQ. Prove that the hyperboloid $\frac{2x^2}{a^2} + \frac{2y^2}{b^2} - \frac{z^2}{c^2} = 1$ is generated by either PQ′ or P′Q.

Ex. 2. A point, "m," on the parabola $y = 0$, $cx^2 = 2a^2z$, is $(2am, 0, 2cm^2)$, and a point, "n," on the parabola $x = 0$, $cy^2 = -2b^2z$, is $(0, 2bn, -2cn^2)$. Find the locus of the lines joining the points for which, (i) $m = n$, (ii) $m = -n$.

Ans. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2z}{c}$.

97. Section of a ruled surface by the tangent plane at a point. Since a hyperboloid of one sheet or a hyperbolic paraboloid is generated completely by each of two systems of straight lines, there pass through any point P, (fig. 43), of the surface, two generating lines, one from each system. Each of these meets the surface at P in, at least, two coincident points, and therefore the lines lie in the tangent

plane at P. The tangent plane at P is therefore the plane through the generators which pass through P. But any plane section of the surface is a conic, and therefore the section of the surface by the tangent plane at P is the conic composed of the two generating lines through P.

It follows that if a straight line AB lies wholly on the conicoid it must belong to one of the systems of generating lines. For AB meets any generating line PQ in some point P, and AB and PQ both lie in the tangent plane at P. But

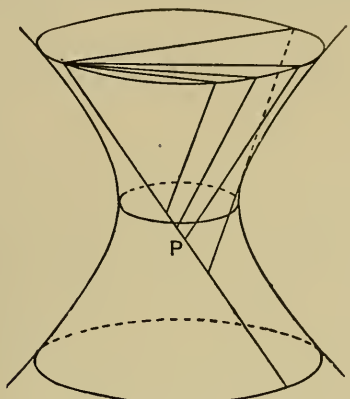

FIG. 43.

the section of the surface by the tangent plane at P consists of the two generators through P, and therefore AB must be one of the generators.

Again any plane through a generating line is the tangent plane at some point of the generating line. For the locus of points common to the surface and plane is a conic, and the generating line is obviously part of the locus. The locus must therefore consist of two straight lines, or the plane must pass through the given generating line and a second generating line which meets it. It is therefore the tangent plane at the point of intersection.

The intersection of a cone or cylinder with a tangent plane consists of two coincident generators. The ruled conicoids can therefore be divided into two classes according

as the generators in which any tangent plane meets them are distinct or coincident. If the generators are distinct the tangent planes at different points of a given generator are different, (see fig. 43). If the generators are coincident, the same plane touches the surface at all points of a given generator.

98. *If three points of a straight line lie on a conicoid the straight line lies wholly on the conicoid.*

The coordinates of any point on the line through (α, β, γ), whose direction-ratios are l, m, n, are $\alpha+lr$, $\beta+mr$, $\gamma+nr$. The condition that this point should lie on the conicoid $F(x, y, z)=0$ may be written, since $F(x, y, z)$ is of the second degree, in the form

$$\mathrm{A}r^2+2\mathrm{B}r+\mathrm{C}=0.$$

If three points of the line lie on the conicoid, this equation is satisfied by three values of r, and therefore $\mathrm{A}=\mathrm{B}=\mathrm{C}=0$. The equation is therefore satisfied by all values of r, and every point of the line lies on the conicoid.

99. *To find the conditions that a given straight line should be a generator of a given conicoid.*

Let the equations to the conicoid and line be

$$ax^2+by^2+cz^2=1,$$

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}.$$

The point on the line, $(\alpha+lr, \beta+mr, \gamma+nr)$, lies on the conicoid if

$$r^2(al^2+bm^2+cn^2)+2r(a\alpha l+b\beta m+c\gamma n)$$
$$+a\alpha^2+b\beta^2+c\gamma^2-1=0.$$

If this equation is an identity, the line lies wholly on the conicoid, and is a generator of the conicoid. The required conditions are therefore

$$a\alpha^2+b\beta^2+c\gamma^2=1, \quad\text{.....................(1)}$$
$$a\alpha l+b\beta m+c\gamma n=0, \quad\text{.....................(2)}$$
$$al^2+bm^2+cn^2=0. \quad\text{.....................(3)}$$

Equation (1) is the condition that (α, β, γ) should lie on the surface; equation (2) shews that a generating line must lie in the tangent plane at any point (α, β, γ) on it; and from (3) it follows that the parallels through the centre to the generating lines generate the asymptotic cone

$$ax^2+by^2+cz^2=0.$$

The three equations (1), (2), (3) shew that through any point (α, β, γ) of a central conicoid two straight lines can be drawn to lie wholly on the conicoid, the direction-ratios of these lines being given by equations (2) and (3). By Lagrange's identity, we have

$$(al^2+bm^2)(a\alpha^2+b\beta^2)-(a\alpha l+b\beta m)^2 \equiv ab(\alpha m-\beta l)^2;$$

whence, by (1), (2), (3),

$$-cn^2=ab(\alpha m-\beta l)^2. \quad \text{.....................(4)}$$

The values of $l:m:n$ are therefore real only if ab and c have opposite signs, which can only be the case if two of the quantities a, b, c are positive and one is negative. The only ruled central conicoid is therefore the hyperboloid of one sheet. From equations (2) and (4) we deduce the direction-ratios of the generators through (α, β, γ),

$$\frac{l}{\pm b\beta\sqrt{\frac{-c}{ab}}+c\alpha\gamma}=\frac{m}{\mp a\alpha\sqrt{\frac{-c}{ab}}+c\beta\gamma}=\frac{n}{-(a\alpha^2+b\beta^2)}.$$

Similarly, the conditions that the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

should be a generator of the paraboloid $ax^2+by^2=2z$ are,

$$a\alpha^2+b\beta^2=2\gamma, \quad \text{.....................(1)}$$

$$a\alpha l+b\beta m-n=0, \quad \text{.....................(2)}$$

$$al^2+bm^2=0. \quad \text{.....................(3)}$$

Equation (3) is satisfied by real values of $l:m$ only if a and b have opposite signs. The only ruled paraboloid

is therefore the hyperbolic paraboloid. The direction-ratios of the generating lines through (α, β, γ) are given by

$$\frac{l}{\sqrt{\frac{1}{a}}}=\frac{m}{\pm\sqrt{\frac{-1}{b}}}=\frac{n}{\alpha\sqrt{a}\pm\beta\sqrt{-b}}.$$

The following examples should be solved in two ways, (i) by factorising the equation to the surface as in § 96, (ii) by means of the conditions in § 99.

Ex. 1. Find the equations to the generating lines of the hyperboloid $\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{16}=1$ which pass through the points $(2, 3, -4)$, $(2, -1, \frac{4}{3})$.

Ans. $\frac{x-2}{1}=\frac{y-3}{0}=\frac{z+4}{-2}$; $\frac{x-2}{0}=\frac{y-3}{3}=\frac{z+4}{-4}$;

$\frac{x-2}{0}=\frac{y+1}{3}=\frac{z-\frac{4}{3}}{-4}$; $\frac{x-2}{3}=\frac{y+1}{6}=\frac{z-\frac{4}{3}}{10}$.

Ex. 2. Find the equations to the generating lines of the hyperboloid $yz+2zx+3xy+6=0$ which pass through the point $(-1, 0, 3)$.

$$[yz+2zx+3xy+6\equiv(y+2)(z+3)+(2z+3y)(x-1).]$$

Ans. $x=-1, z=3$; $\frac{x+1}{1}=\frac{y}{-1}=\frac{z-3}{3}$.

Ex. 3. Find the equations to the generators of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ which pass through the point $(a\cos\alpha, b\sin\alpha, 0)$.

Ans. $\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{-b\cos\alpha}=\frac{z}{\pm c}$.

Ex. 4. Find the equations to the generating lines of the paraboloid $(x+y+z)(2x+y-z)=6z$ which pass through the point $(1, 1, 1)$.

Ans. $\frac{x-1}{1}=\frac{y-1}{-3}=\frac{z-1}{-1}$; $\frac{x-1}{4}=\frac{y-1}{-5}=\frac{z-1}{1}$.

THE SYSTEMS OF GENERATING LINES.

100. We shall call the systems of generating lines of the hyperboloid of one sheet which are given by the equations

$$\frac{x}{a}+\frac{z}{c}=\lambda\left(1+\frac{y}{b}\right),\quad \frac{x}{a}-\frac{z}{c}=\frac{1}{\lambda}\left(1-\frac{y}{b}\right); \quad\text{............(1)}$$

$$\frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right),\quad \frac{x}{a}+\frac{z}{c}=\frac{1}{\mu}\left(1-\frac{y}{b}\right), \quad\text{............(2)}$$

the λ-system and μ-system, respectively.

101. *No two generators of the same system intersect.*

For the equations (1) and

$$\frac{x}{a}+\frac{z}{c}=\lambda'\left(1+\frac{y}{b}\right),\quad \frac{x}{a}-\frac{z}{c}=\frac{1}{\lambda'}\left(1-\frac{y}{b}\right)$$

lead to $\quad \frac{x}{a}+\frac{z}{c}=0,\quad \frac{x}{a}-\frac{z}{c}=0,\quad 1+\frac{y}{b}=0,\quad 1-\frac{y}{b}=0,$

which are obviously inconsistent.

Otherwise, if P and Q are any points on any generator of the μ-system and the generators of the λ-system through P and Q intersect at R, then the plane PQR meets the hyperboloid in the sides of a triangle. This is impossible, since no plane section of a conicoid is of higher degree than the second.

102. *Any generator of the λ-system intersects any generator of the μ-system.*

From the equations (1) and (2),

$$\frac{\frac{x}{a}+\frac{z}{c}}{\lambda}=\frac{\frac{x}{a}-\frac{z}{c}}{\mu}=\frac{1-\frac{y}{b}}{\lambda\mu}=\frac{1+\frac{y}{b}}{1}.$$

Whence, adding and subtracting numerators and denominators,

$$\frac{x}{a}=\frac{\lambda+\mu}{1+\lambda\mu},\quad \frac{y}{b}=\frac{1-\lambda\mu}{1+\lambda\mu},\quad \frac{z}{c}=\frac{\lambda-\mu}{1+\lambda\mu}.$$

These determine the point of intersection.

The equations

$$\frac{x}{a}+\frac{z}{c}-\lambda\left(1+\frac{y}{b}\right)+k\left\{\frac{x}{a}-\frac{z}{c}-\frac{1}{\lambda}\left(1-\frac{y}{b}\right)\right\}=0, \quad\ldots\ldots(3)$$

$$\frac{x}{a}-\frac{z}{c}-\mu\left(1+\frac{y}{b}\right)+k'\left\{\frac{x}{a}+\frac{z}{c}-\frac{1}{\mu}\left(1-\frac{y}{b}\right)\right\}=0 \quad\ldots\ldots(4)$$

both reduce to

$$\frac{x}{a}(\lambda+\mu)+\frac{y}{b}(1-\lambda\mu)-\frac{z}{c}(\lambda-\mu)=1+\lambda\mu, \quad\ldots\ldots\ldots(5)$$

if $k=1/k'=\lambda/\mu$. But equation (5) represents the tangent plane at the point of intersection of the generators. Hence the plane through two intersecting generators is the tangent plane at their common point. (Cf. § 97.)

If, in equation (3), k is given, the equation represents

a given plane through the generator. But the equation reduces to equation (5) if $\mu=\lambda/k$. Hence any plane through a generating line is a tangent plane.

Ex. Discuss the intersection of the λ-generator through P with the μ-generator through P′ when P and P′ are the extremities of a diameter of the principal elliptic section.

103. Perpendicular generators. *To find the locus of the points of intersection of perpendicular generators.*

The direction-cosines of the λ- and μ-generators are given by, (§ 42),

$$\frac{l/a}{\lambda^2-1}=\frac{m/b}{2\lambda}=\frac{n/c}{\lambda^2+1}, \qquad \frac{l/a}{\mu^2-1}=\frac{m/b}{2\mu}=\frac{n/c}{-(\mu^2+1)}.$$

The condition that the generators should be at right angles is

$$a^2(\lambda^2-1)(\mu^2-1)+4b^2\lambda\mu-c^2(\lambda^2+1)(\mu^2+1)=0,$$

which may be written

$$a^2(\lambda+\mu)^2+b^2(1-\lambda\mu)^2+c^2(\lambda-\mu)^2=(a^2+b^2-c^2)(1+\lambda\mu)^2,$$

and shews that their point of intersection

$$\left\{\frac{a(\lambda+\mu)}{1+\lambda\mu}, \ \frac{b(1-\lambda\mu)}{1+\lambda\mu}, \ \frac{c(\lambda-\mu)}{1+\lambda\mu}\right\}$$

lies on the director sphere

$$x^2+y^2+z^2=a^2+b^2-c^2.$$

The locus is therefore the curve of intersection of the hyperboloid and the director sphere.

Or if PQ, PR are perpendicular generators and PN is normal at P, by § 102 the planes PQR, PNQ, PNR are mutually perpendicular tangent planes, and therefore P lies on the director sphere.

104. *The projections of the generators of a hyperboloid on a principal plane are tangents to the section of the hyperboloid by the principal plane.*

The projections of the λ- and μ-generators on the plane XOY are given by

$$z=0, \quad \frac{2x}{a}=\lambda\left(1+\frac{y}{b}\right)+\frac{1}{\lambda}\left(1-\frac{y}{b}\right);$$

$$z=0, \quad \frac{2x}{a}=\mu\left(1+\frac{y}{b}\right)+\frac{1}{\mu}\left(1-\frac{y}{b}\right),$$

which may be written

$$z=0, \quad \lambda^2\left(1+\frac{y}{b}\right)-\frac{2\lambda x}{a}+\left(1-\frac{y}{b}\right)=0, \quad \text{etc.}$$

Whence the envelope of the projections is the ellipse

$$z=0, \quad \frac{x^2}{a^2}=1-\frac{y^2}{b^2}.$$

Similarly, the projections on the planes YOZ, ZOX touch the corresponding principal sections.

The above equations to the projections are identical if $\lambda=\mu$. Hence equal values of the parameters give two generators which project into the same tangent to the ellipse $z=0$, $x^2/a^2+y^2/b^2=1$. The point of intersection, P, of the generators given by $\lambda=\mu=t$ is, by § 102,

$$\left(a\frac{2t}{1+t^2}, \quad b\frac{1-t^2}{1+t^2}, \quad 0\right),$$

i.e. is $(a\cos\alpha, b\sin\alpha, 0)$, where $t\equiv\tan\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$. P is therefore the point on the principal elliptic section whose eccentric angle is α, and the generators project into the tangent

$$z=0, \quad \frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=1, \quad \text{(fig. 44)}.$$

From § 103, the direction-cosines of the λ-generator are proportional to

$$a\frac{\lambda^2-1}{\lambda^2+1}, \quad b\frac{2\lambda}{\lambda^2+1}, \quad c,$$

or, since $\lambda=\tan\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$, $a\sin\alpha$, $-b\cos\alpha$, $-c$.

Therefore the equations to the λ-generator through P are

$$\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{-b\cos\alpha}=\frac{z}{-c}.$$

Similarly, the equations to the μ-generator are

$$\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{-b\cos\alpha}=\frac{z}{c}.$$

Ex. Prove that the generators given by $\lambda=t$, $\mu=-1/t$ are parallel, and that they meet the principal elliptic section in the extremities of a diameter.

105. Let P and Q, (fig. 44), be the points on the principal elliptic section whose eccentric angles are α and β. Then the λ-generator through P intersects the μ-generator through Q at the point, R,

$$\left(a\frac{\lambda+\mu}{1+\lambda\mu},\quad b\frac{1-\lambda\mu}{1+\lambda\mu},\quad c\frac{\lambda-\mu}{1+\lambda\mu}\right),$$

where $\lambda \equiv \tan\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$ and $\mu \equiv \tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)$.

Whence the coordinates of R are

$$a\frac{\cos\frac{\beta+\alpha}{2}}{\cos\frac{\beta-\alpha}{2}},\quad b\frac{\sin\frac{\beta+\alpha}{2}}{\cos\frac{\beta-\alpha}{2}},\quad c\tan\frac{\beta-\alpha}{2}.$$

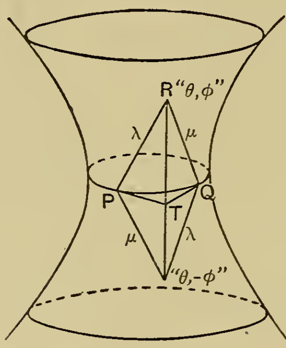

FIG. 44.

Now the coordinates of any point on the hyperboloid can be expressed in the form

$$a\cos\theta\sec\phi,\quad b\sin\theta\sec\phi,\quad c\tan\phi;$$

therefore if R is the point "θ, ϕ,"

$$\theta=\frac{\beta+\alpha}{2} \text{ and } \phi=\frac{\beta-\alpha}{2},$$

or $$\theta+\phi=\beta,\quad \theta-\phi=\alpha.$$

Similarly, it may be shewn that the μ-generator through P intersects the λ-generator through Q at the point "$\theta, -\phi$."

Suppose now that P remains fixed while Q varies, so that α is constant and β variable. R then moves along a fixed

generator of the λ-system, and in any position $\theta-\phi=\alpha$. Hence for points on a generator of the λ-system $\theta-\phi$ is constant. Similarly, by supposing Q to remain fixed and P to vary, we can prove that for points on a given generator of the μ-system $\theta+\phi$ is constant.

Ex. 1. If R is "θ, ϕ," (fig. 44), shew that the equations to PQ are $z=0$, $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=\cos\phi$, and deduce that $\theta-\phi=\alpha$, $\theta+\phi=\beta$.

Ex. 2. The equations to the generating lines through "θ, ϕ" are

$$\frac{x-a\cos\theta\sec\phi}{a\sin(\theta\pm\phi)}=\frac{y-b\sin\theta\sec\phi}{-b\cos(\theta\pm\phi)}=\frac{z-c\tan\phi}{\pm c}.$$

Ex. 3. If $(a\cos\theta\sec\phi,\ b\sin\theta\sec\phi,\ c\tan\phi)$ is a point on the generating line

$$\frac{x}{a}+\frac{z}{c}=\lambda\left(1+\frac{y}{b}\right),\quad \frac{x}{a}-\frac{z}{c}=\frac{1}{\lambda}\left(1-\frac{y}{b}\right),$$

prove that $\tan\frac{\theta-\phi}{2}=\frac{1-\lambda}{1+\lambda}$, and hence shew that for points of a given generator of the λ-system $\theta-\phi$ is constant.

Ex. 4. Prove that the equations

$$\frac{x}{a}=\frac{\cos(\theta-\phi)}{\cos(\theta+\phi)},\quad \frac{y}{b}=\frac{\cos\theta\sin\phi}{\cos(\theta+\phi)},\quad \frac{z}{c}=\frac{\sin\theta\cos\phi}{\cos(\theta+\phi)}$$

determine a hyperboloid of one sheet, that θ is constant for points on a given generator of one system, and that ϕ is constant for points on a given generator of the other system.

$\left(\text{The equation to the surface is } \frac{x^2}{a^2}-\frac{4yz}{bc}=1.\right)$

Ex. 5. Find the locus of R if P and Q are the extremities of conjugate diameters of the principal elliptic section.

We have $\theta-\phi=\alpha$, $\theta+\phi=\alpha\pm\frac{\pi}{2}$, whence $\phi=\pm\frac{\pi}{4}$, and R lies in one of the planes $z=\pm c$.

Ex. 6. Prove also that $RP^2+RQ^2=a^2+b^2+2c^2$.

Ex. 7. If A and A′ are the extremities of the major axis of the principal elliptic section, and any generator meets two generators of the same system through A and A′ respectively in P and P′, prove that $AP.A'P'=b^2+c^2$.

Ex. 8. Prove also that the planes APP′, A′PP′ cut either of the real central circular sections in perpendicular lines.

Ex. 9. If four generators of the hyperboloid form a skew quadrilateral whose vertices are "θ_r, ϕ_r," $r=1, 2, 3, 4$, prove that

$$\theta_1+\theta_3=\theta_2+\theta_4,\quad \phi_1+\phi_3=\phi_2+\phi_4.$$

Ex. 10. Interpret the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=\left(\frac{xx'}{a^2}+\frac{yy'}{b^2}-\frac{zz'}{c^2}\right)^2,$$

where P, (x', y', z') is a point on the hyperboloid.

[The equation represents the pair of planes through the origin and the generators that intersect at (x', y', z').]

Ex. 11. Prove that the generators through any point P on a hyperboloid are parallel to the asymptotes of the section of the hyperboloid by any plane which is parallel to the tangent plane at P.

Ex. 12. Prove that the angle between the generators through P is given by

$$\cot\theta=\frac{p(r^2-a^2-b^2+c^2)}{2abc},$$

where p is the perpendicular from the centre to the tangent plane at P and r is the distance of P from the centre.

Ex. 13. All parallelepipeds which have six of their edges along generators of a given hyperboloid have the same volume.

If PQRS is one face of the parallelepiped and P, P′; Q, Q′; R, R′; S, S′ are opposite corners, we may have the edges PS, RP′, S′R′ along generators of one system and the edges SR, P′S′, R′P along generators of the other system. The tangent planes at S and S′ are therefore PSR, P′S′R′, and are parallel, and therefore SS′ is a diameter. Similarly, PP′ and RR′ are diameters. Let P, S, R be (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3). Then the volume of the parallelepiped is twelve times the volume of the tetrahedron OPSR, (O is the centre). Denoting it by V, we have

$$V=2\begin{vmatrix} x & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}=2abc\sqrt{-1}\begin{vmatrix} \frac{x_1}{a}, & \frac{y_1}{b}, & \frac{z_1}{\sqrt{-c^2}} \\ \frac{x_2}{a}, & \frac{y_2}{b}, & \frac{z_2}{\sqrt{-c^2}} \\ \frac{x_3}{a}, & \frac{y_3}{b}, & \frac{z_3}{\sqrt{-c^2}} \end{vmatrix}$$

$$=2abc\sqrt{-1}\begin{vmatrix} \Sigma\frac{x_1^2}{a^2}, & \Sigma\frac{x_1x_2}{a^2}, & \Sigma\frac{x_1x_3}{a^2} \\ \Sigma\frac{x_1x_2}{a^2}, & \Sigma\frac{x_2^2}{a^2}, & \Sigma\frac{x_2x_3}{a^2} \\ \Sigma\frac{x_1x_3}{a^2}, & \Sigma\frac{x_2x_3}{a^2}, & \Sigma\frac{x_3^2}{a^2} \end{vmatrix}^{\frac{1}{2}}.$$

But
$$\Sigma\frac{x_1^2}{a^2}\equiv\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-\frac{z_1^2}{c^2}=1,\quad \Sigma\frac{x_2^2}{a^2}=1,\quad \Sigma\frac{x_3^2}{a^2}=1;$$

and, since R′ and S are on the tangent plane at P, and S on the tangent plane at R,

$$\Sigma\frac{x_1x_2}{a^2}=1,\quad \Sigma\frac{x_2x_3}{a^2}=1,\quad \Sigma\frac{x_1x_3}{a^2}=-1.$$

Therefore
$$V=2abc\sqrt{-1}(-4)^{\frac{1}{2}}=4abc.$$

Ex. 14. Find the locus of the corners Q and Q′ which are not on the given hyperboloid.

Since QS and PR bisect one another, Q is the point

$$(x_1-x_2+x_3,\ \ y_1-y_2+y_3,\ \ z_1-z_2+z_3),$$

and hence lies on the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}+3=0.$$

106. The systems of generators of the hyperbolic paraboloid. We shall now state the results for the hyperbolic paraboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$ corresponding to those which we have proved for the hyperboloid. Their proof is left as an exercise for the student.

The point of intersection of the generators

$$\frac{x}{a}-\frac{y}{b}=2\lambda,\quad \frac{x}{a}+\frac{y}{b}=\frac{z}{\lambda};\ \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\frac{x}{a}+\frac{y}{b}=2\mu,\quad \frac{x}{a}-\frac{y}{b}=\frac{z}{\mu}\ \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

is given by

$$\frac{x}{a}=\mu+\lambda,\quad \frac{y}{b}=\mu-\lambda,\quad z=2\lambda\mu.$$

The direction-cosines of the generators are given by

$$\frac{l}{a}=\frac{m}{b}=\frac{n}{2\lambda};\quad \frac{l}{a}=\frac{m}{-b}=\frac{n}{2\mu},$$

and hence the locus of the points of intersection of perpendicular generators is the curve of intersection of the surface and the plane $2z+a^2-b^2=0$.

The plane

$$\frac{x}{a}-\frac{y}{b}-2\lambda+k\left\{\frac{x}{a}+\frac{y}{b}-\frac{z}{\lambda}\right\}=0$$

passes through the generator (1) and is tangent plane at the point of intersection of that generator and the generator of the μ-system given by $\mu=\lambda/k$.

The projections of the generators on the planes YOZ, ZOX envelope the principal sections whose equations are

$$x=0,\ y^2=-2b^2z;\quad y=0,\ x^2=2a^2z.$$

Any point on the second parabola is $(2am, 0, 2m^2)$, and if $\lambda=\mu=m$, the generators of the λ- and μ-systems corresponding to these values project into the tangent to the parabola at "m."

Any point on the surface is given by

$$x=ar\cos\theta,\quad y=br\sin\theta,\quad 2z=r^2\cos 2\theta,$$

and the equations to the generators through "r, θ" are

$$\frac{x-ar\cos\theta}{a}=\frac{y-br\sin\theta}{\pm b}=\frac{z-\dfrac{r^2}{2}\cos 2\theta}{r(\cos\theta\mp\sin\theta)}.$$

Ex. 1. Shew that the angle between the generating lines through (x, y, z) is given by

$$\tan\theta=ab\left(1+\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)^{\frac{1}{2}}\left(z+\frac{a^2-b^2}{2}\right)^{-1}.$$

Ex. 2. Prove that the equations

$$4x=a(1+\cos 2\theta),\quad y=b\cosh\phi\cos\theta,\quad z=c\sinh\phi\cos\theta$$

determine a hyperbolic paraboloid, and that the angle between the generators through "θ, ϕ" is given by

$$\sec\psi=\frac{\{(b^2+c^2)^2+a^4\cos^4\theta+2a^2(b^2+c^2)\cos^2\theta\cosh 2\phi\}^{\frac{1}{2}}}{b^2-c^2+a^2\cos^2\theta}.$$

Ex. 3. Prove that the equations

$$2x=ae^{2\phi},\quad y=be^{\phi}\cosh\theta,\quad z=ce^{\phi}\sinh\theta$$

determine a hyperbolic paraboloid, and that $\theta+\phi$ is constant for points of a given generator of one system, and $\theta-\phi$ is constant for a given generator of the other.

Ex. 4. Planes are drawn through the origin, O, and the generators through any point P of the paraboloid given by $x^2-y^2=az$. Prove that the angle between them is $\tan^{-1}\dfrac{2r}{a}$, where r is the length of OP.

Ex. 5. Find the locus of the perpendiculars from the vertex of the paraboloid $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=2z$ to the generators of one system.

Ans. $x^2+y^2+2z^2\pm\dfrac{a^2+b^2}{ab}xy=0.$

Ex. 6. The points of intersection of generators of $xy=az$ which are inclined at a constant angle α lie on the curve of intersection of the paraboloid and the hyperboloid $x^2+y^2-z^2\tan^2\alpha+a^2=0.$

107. Conicoids through three given lines. The general equation to a conicoid,

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0,$$

contains nine constants, viz., the ratios of any nine of the ten coefficients a, b, c, ... to the tenth. Hence, since these are determined by nine equations involving them, a conicoid can be found to pass through nine given points. But we have proved that if three points of a straight line lie on a given conicoid, the line is a generator of the conicoid. Therefore a conicoid can be found to pass through any three given non-intersecting lines.

108. The general equation to a conicoid through the two given lines $u=0=v$, $u'=0=v'$, is

$$\lambda uu'+\mu uv'+\nu vu'+\rho vv'=0,$$

since this equation is satisfied when $u=0$ and $v=0$, or when $u'=0$ and $v'=0$, and contains three disposable constants, viz. the ratios of λ, μ, ν to ρ.

109. *To find the equation to the conicoid through three given non-intersecting lines.*

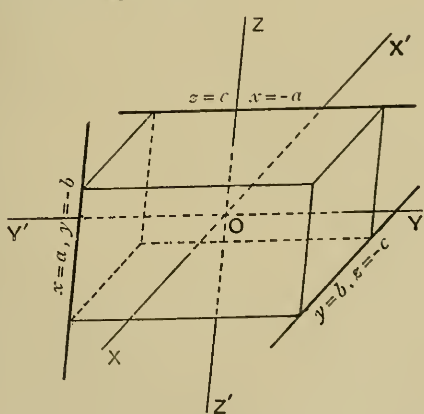

FIG. 45.

If the three lines are not parallel to the same plane, planes drawn through each line parallel to the other two form a parallelepiped, (fig. 45). If the centre of the

parallelepiped is taken as origin, and the axes are parallel to the edges, the equations to the given lines are of the form,

(1) $y=b,\ z=-c$; (2) $z=c,\ x=-a$; (3) $x=a,\ y=-b$,

where $2a$, $2b$, $2c$ are the edges. The general equation to a conicoid through the lines (1) and (2) is

$$(y-b)(z-c)+\lambda(y-b)(x+a)$$
$$+\mu(z+c)(z-c)+\nu(z+c)(x+a)=0.$$

Where $x=a,\ y=-b$ meets the surface we have

$$\mu z^2+2z(a\nu-b)-\mu c^2+2c(a\nu+b)-4ab\lambda=0,$$

and if $x=a,\ y=-b$ is a generator, this equation must be satisfied for *all* values of z. Therefore

$$\mu=0, \quad \nu=\frac{b}{a}, \quad \lambda=\frac{c(a\nu+b)}{2ab}=\frac{c}{a},$$

and the equation to the surface is

$$ayz+bzx+cxy+abc=0.$$

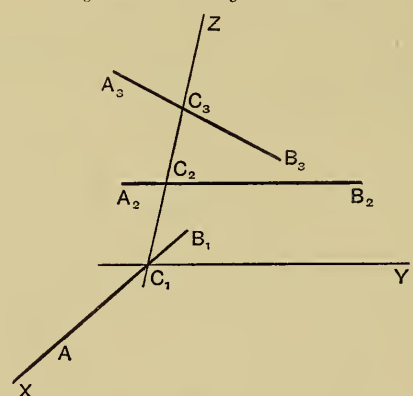

FIG. 46.

The origin evidently bisects all chords of the surface which pass through it, and therefore the surface is a central surface, and is therefore a hyperboloid of one sheet. (Cf. § 47, Ex. 1.)

If the three lines are parallel to the same plane, let any line which meets them be taken as z-axis. If the lines are A_1B_1, A_2B_2, A_3B_3, (fig. 46), and the z-axis meets them in

C_1, C_2, C_3, take A_1B_1 as x-axis and the parallel to A_2B_2 through C_1 as y-axis. Then the equations may be written

(1) $y=0$, $z=0$; (2) $x=0$, $z=\alpha$; (3) $lx+my=0$, $z=\beta$.

The equation to a conicoid through the lines (2) and (3) is

$$\lambda x(lx+my)+\mu x(z-\beta)$$
$$+\nu(z-\alpha)(lx+my)+\rho(z-\alpha)(z-\beta)=0.$$

If $y=0$, $z=0$ is a generator, the equation

$$l\lambda x^2-x(\mu\beta+\nu l\alpha)+\rho\alpha\beta=0$$

must be satisfied for all values of x, and therefore

$$\lambda=\rho=0, \quad \mu\beta+\nu l\alpha=0;$$

and hence the equation to the surface is

$$z\{lx(\alpha-\beta)-\beta my\}+\alpha\beta my=0.$$

Since the terms of second degree are the product of linear factors, the equation represents a hyperbolic paraboloid.

110. The straight lines which meet four given lines. If A, B, C are three given non-intersecting lines, an infinite number of straight lines can be drawn to meet A, B, and C. For a conicoid can be drawn through A, B, C, and A, B, C are generators of one system, say the λ-system, and hence all the generators of the μ-system will intersect A, B, and C.

A fourth line, D, which does not meet A, B, and C, meets the conicoid in general in two points P and Q, and the generators of the μ-system through P and Q are the only lines which intersect the four given lines A, B, C, D. If, however, D is a generator of the conicoid through A, B, and C, it belongs to the λ-system, and therefore all the generators of the μ-system meet all the four lines.

111. *If three straight lines can be drawn to meet four given non-intersecting lines* A, B, C, D, *then* A, B, C, D *are generators of a conicoid.*

If the three lines are P, Q, R, each meets the conicoid through A, B, C in three points, and is therefore a generator. Hence D meets the conicoid in three points, viz. the points of intersection of D and P, Q, R; and therefore D is a generator.

Ex. 1. A, A′; B, B′; C, C′ are points on X′OX, Y′OY, Z′OZ. Prove that BC′, CA′, AB′ are generators of one system, and that B′C, C′A, A′B are generators of the other system, of a hyperboloid.

Ex. 2. A, A′; B, B′; C, C′ are pairs of opposite vertices of a skew hexagon drawn on a hyperboloid. Prove that AA′, BB′, CC′ are concurrent.

Ex. 3. The altitudes of a tetrahedron are generators of a hyperboloid of one sheet.

Let A, B, C, D be the vertices. Then the planes through DA, perpendicular to the plane DBC, through DB, perpendicular to the plane DCA, and through DC, perpendicular to the plane DAB, pass through one line, (§ 45, Ex. 6, or § 44, Ex. 22). That line is therefore coplanar with the altitudes from A, B, C, and it meets the altitude from D in D, and therefore it meets all the four altitudes. The corresponding lines through A, B, C also meet all the four altitudes, which are therefore generators of a hyperboloid.

Ex. 4. Prove that the perpendiculars to the faces of the tetrahedron through their orthocentres are generators of the opposite system.

Ex. 5. Prove that the lines joining A, B, C, D to the centres of the circles inscribed in the triangles BCD, CDA, DAB, ABC are generators of a hyperboloid.

112. The equation to a hyperboloid when two intersecting generators are coordinate axes. If two intersecting generators are taken as x-axis and y-axis, the equation to the surface must be satisfied by all values of x when $y=z=0$, and by all values of y when $z=x=0$.

Suppose that it is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz=0.$$

Then we must have

$$a=u=0, \text{ and } b=v=0,$$

and therefore the equation takes the form

$$cz^2+2fyz+2gzx+2hxy+2wz=0.$$

Suppose now that the line joining the point of intersection of the generators to the centre is taken as z-axis. Then, since the generators through opposite ends of a diameter are parallel, the lines $y=0$, $z=2\gamma$; $x=0$, $z=2\gamma$ are generators, the centre being $(0, 0, \gamma)$. Whence

$$f=g=0, \quad \gamma=-w/c,$$

and the equation reduces to

$$cz^2+2hxy+2wz=0.$$

Ex. 1. Prove that $(y+mz)(z+nx)+kz=0$ represents a paraboloid which passes through OX and OY.

Ex. 2. The generators through a variable point P of a hyperboloid meet the generators through a fixed point O in Q and R. If OQ : OR is constant, find the locus of P.

Take OQ and OR as x- and y-axes, and the line joining O to the centre as z-axis. The equation to the hyperboloid is

$$cz^2+2hxy+2wz=0.$$

It may be written $\quad z(cz+2w)+2hxy=0,$

and hence the systems of generating lines are given by

$$z=2h\lambda x, \quad \lambda(cz+2w)+y=0\,;$$

$$z=2h\mu y, \quad \mu(cz+2w)+x=0.$$

OX belongs to the λ-system and corresponds to $\lambda=0$; OY belongs to the μ-system and corresponds to $\mu=0$. If P is (ξ, η, ζ), the generators through P correspond to

$$\lambda=\zeta/2h\xi, \quad \mu=\zeta/2h\eta.$$

Where a generator of the μ-system meets OX,

$$y=0, \quad z=0, \quad x=-2w\mu,$$

therefore $\quad \mathrm{OQ}=-2w\mu=-w\zeta/h\eta.$

Similarly, $\quad \mathrm{OR}=-2w\lambda=-w\zeta/h\xi,$

and P therefore lies on the plane $x=ky$.

[OQ and OR may be found more easily by considering that the plane PQR is the tangent plane at P whose equation (see § 134) is $h\eta x+h\xi y+(c\zeta+w)z+w\zeta=0$.]

Ex. 3. Find the locus of P if (i) $\mathrm{OQ}\cdot\mathrm{OR}=k^2$, (ii) $\mathrm{OQ}^2+\mathrm{OR}^2=k^2$.

Ex. 4. If $\mathrm{OQ}^{-2}+\mathrm{OR}^{-2}$ is constant, P lies on a cone whose vertex is O and whose section by a plane parallel to OXY is an ellipse whose equal conjugate diameters are parallel to OX and OY.

Ex. 5. Shew that the projections of the generators of one system of a hyperboloid on the tangent plane at any point envelope a conic.

Take the generators in the given tangent plane as OX and OY, and the normal at O as OZ. The plane $z=\lambda y$ is a tangent plane, (§ 97), and the projection on OXY of the second generator in which it meets the surface has equations

$$z=0, \quad c\lambda^2y+2\lambda(gx+fy+w)+2hx=0.$$

Whence the envelope of the projections is the conic

$$z=0, \quad (gx+fy+w)^2=2chxy$$

***113. Properties of a given generating line.** If we have a system of rectangular axes in which the x-axis is a generator and the z-axis is the normal at the origin, the equation to the hyperboloid is of the form,

$$by^2+cz^2+2fyz+2gzx+2hxy+2wz=0,$$

or $\quad y(by+2hx)+z(cz+2gx+2fy+2w)=0.$

The systems of generating lines are given by

$$\lambda y = z, \quad (by + 2hx) + \lambda(cz + 2gx + 2fy + 2w) = 0;$$
$$y = \mu(cz + 2gx + 2fy + 2w), \quad z + \mu(by + 2hx) = 0.$$

The x-axis belongs to the μ-system and corresponds to $\mu = 0$. The generator of the λ-system through the point $(\alpha, 0, 0)$ is given by

$$\lambda = \frac{-h\alpha}{g\alpha + w}.$$

The tangent plane at $(\alpha, 0, 0)$ is the plane through this generator and OX. Its equation is therefore $\lambda y = z$,

or
$$h\alpha y + z(g\alpha + w) = 0.$$

Let P, $(\alpha, 0, 0)$, P′, $(\alpha', 0, 0)$ be points on the x-axis. Then the tangent planes at P and P′ are at right angles if

$$(h^2 + g^2)\alpha\alpha' + wg(\alpha + \alpha') + w^2 = 0,$$

$$\textit{i.e.} \text{ if } \left(\alpha + \frac{wg}{h^2 + g^2}\right)\left(\alpha' + \frac{wg}{h^2 + g^2}\right) = \frac{-w^2h^2}{(h^2 + g^2)^2}. \quad \ldots\ldots(1)$$

Therefore if C is the point $\left(-\frac{wg}{h^2 + g^2}, 0, 0\right)$, CP · CP′ is constant for all pairs of perpendicular tangent planes through OX. C is called the **central point** of the generator OX. If the origin is taken at the central point, the equation (1) must take the form $\alpha\alpha' = \text{constant}$, and therefore $g = 0$, and $\alpha\alpha' = -w^2/h^2$. The equation to the conicoid when OX is a generator and O is the central point, OZ is the normal at O, and the axes are rectangular, is therefore

$$by^2 + cz^2 + 2fyz + 2hxy + 2wz = 0.$$

Ex. 1. Find the locus of the normals to a conicoid at points of a given generator.

Taking axes as above, the equations to the normal at $(\alpha, 0, 0)$ are $\frac{x - \alpha}{0} = \frac{y}{h\alpha} = \frac{z}{w}$. The locus of the normals is therefore the hyperbolic paraboloid whose equation is $hxz = wy$. It has OX and OZ as generators, and its vertex at the origin.

Ex. 2. The anharmonic ratio of four tangent planes through the same generator is the anharmonic ratio of their points of contact.

The tangent plane at $(\alpha_r, 0, 0)$ is $h\alpha_r y + wz = 0$, whence, by § 38, Ex. 4, the anharmonic ratio of the planes is

$$\frac{(\alpha_1 - \alpha_2)(\alpha_3 - \alpha_4)}{(\alpha_3 - \alpha_2)(\alpha_1 - \alpha_4)}.$$

Ex. 3. Four fixed generators of the same system meet any generator of the opposite system in a range of constant anharmonic ratio.

Ex. 4. Find the locus of the perpendiculars from a point on a hyperboloid to the generators of one system.

Take O, the point, as origin, and a generator through O as OX. Take the normal at O as OZ, then XOY is the tangent plane at O. The equation to the hyperboloid is

$$by^2+cz^2+2fyz+2gzx+2hxy+2wz=0.$$

The systems of generators are given by

$$\lambda y=z, \quad (by+2hx)+\lambda(cz+2gx+2fy+2w)=0\,;$$
$$y=\mu(cz+2gx+2fy+2w), \quad z+\mu(by+2hx)=0.$$

The locus of the perpendiculars to the generators of the λ-system is the cubic cone

$$x(cz^2+2fyz+by^2)-2(y^2+z^2)(hy+gz)=0.$$

***114. The central point and parameter of distribution.** Taking the axes indicated in § 113 the equation to the conicoid is
$$by^2+cz^2+2fyz+2hxy+2wz=0.$$

The equations to the system of generators to which OX belongs are

$$y=\lambda(2fy+cz+2w), \quad z+\lambda(2hx+by)=0,$$

OX being given by $\lambda=0$. The direction-cosines of a generator of this system are proportional to

$$bc\lambda^2-2f\lambda+1, \quad -2ch\lambda^2, \quad 2h\lambda(2f\lambda-1),$$

and therefore the shortest distance between this generator and OX has direction-cosines proportional to

$$0, \quad 2f\lambda-1, \quad c\lambda.$$

Hence the limiting position of the shortest distance, as λ tends to zero, is parallel to OY. Again, any plane through the generator is given by

$$y(2f\lambda-1)+c\lambda z+2w\lambda-k\{2h\lambda x+b\lambda y+z\}=0.$$

This plane meets OX where $x=w/hk$. It contains the S.D. if
$$(2f\lambda-1)(2f\lambda-1-bk\lambda)+c\lambda(c\lambda-k)=0,$$

i.e. if
$$k=\frac{1-4f\lambda}{(c-b)\lambda},$$

squares and higher powers of λ being rejected.

Therefore the S.D. meets OX where

$$x=\frac{w(c-b)\lambda}{h(1-4f\lambda)}.$$

Since x tends to zero with λ, the limiting position of the S.D. is OY. Hence the central point of a given generator is the point of intersection of the generator and the shortest distance between it and a consecutive generator of the same system.

The equation $y=\lambda(2fy+cz+2w)$ represents the plane through the λ-generator parallel to OX. Therefore the shortest distance, δ, is given by

$$\delta=\frac{2w\lambda}{\sqrt{(1-2f\lambda)^2+c^2\lambda^2}}=2w\lambda,$$

rejecting λ^2, etc.

Again, if θ is the angle between the generator and OX,

$$\cos\theta=\frac{bc\lambda^2-2f\lambda+1}{\sqrt{(bc\lambda^2-2f\lambda+1)^2+4c^2h^2\lambda^4+4h^2\lambda^2(2f\lambda-1)^2}},$$

whence, if λ^2 and higher powers be rejected,

$$\theta=2h\lambda.$$

The limit of the ratio δ/θ, as λ tends to zero, is called the **parameter of distribution** of the generator OX. Denoting it by p, we have

$$p=\text{Lt}\frac{2w\lambda}{2h\lambda}=\frac{w}{h}.$$

Cor. If O is the central point and the tangent planes at A and A′ are at right angles, OA . OA′ $=-p^2$.

Ex. 1. If the generator "ϕ" of the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

is given by
$$\frac{x-a\cos\phi}{a\sin\phi}=\frac{y-b\sin\phi}{-b\cos\phi}=\frac{z}{-c},$$

and θ is the angle between the generators "ϕ" and "ϕ_1," prove that

$$\sin^2\theta=\frac{a^2b^2\sin^2(\phi-\phi_1)+a^2c^2(\sin\phi-\sin\phi_1)^2+b^2c^2(\cos\phi-\cos\phi_1)^2}{(a^2\sin^2\phi+b^2\cos^2\phi+c^2)(a^2\sin^2\phi_1+b^2\cos^2\phi_1+c^2)},$$

and deduce that $$\frac{d\theta}{d\phi}=\frac{(a^2b^2+b^2c^2\sin^2\phi+c^2a^2\cos^2\phi)^{\frac{1}{2}}}{a^2\sin^2\phi+b^2\cos^2\phi+c^2}.$$

Ex. 2. Prove that the shortest distance, δ, between the generators "ϕ" and "ϕ_1" is given by

$$\delta=\frac{2abc\sin\dfrac{\phi_1-\phi}{2}}{\left(b^2c^2\sin^2\dfrac{\phi_1+\phi}{2}+c^2a^2\cos^2\dfrac{\phi_1+\phi}{2}+a^2b^2\cos^2\dfrac{\phi_1-\phi}{2}\right)^{\frac{1}{2}}}$$

and deduce that $\dfrac{d\delta}{d\phi}=\dfrac{abc}{(a^2b^2+b^2c^2\sin^2\phi+c^2a^2\cos^2\phi)^{\frac{1}{2}}}$.

Ex. 3. Prove that the parameter of distribution for the generator "ϕ" is

$$\frac{abc(a^2\sin^2\phi+b^2\cos^2\phi+c^2)}{a^2b^2+b^2c^2\sin^2\phi+c^2a^2\cos^2\phi}.$$

Ex. 4. If D is the distance of any generator of the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

from the centre, and p is its parameter of distribution, $D^2p=abc$.

Ex. 5. Find the coordinates of the central point of the generator "ϕ."

The equation to the plane through the generator "ϕ" parallel to the generator "ϕ_1" is

$$\frac{x}{a}\sin\frac{\phi+\phi_1}{2}-\frac{y}{b}\cos\frac{\phi+\phi_1}{2}+\frac{z}{c}\cos\frac{\phi-\phi_1}{2}+\sin\frac{\phi-\phi_1}{2}=0.$$

Whence the direction-cosines of the S.D. between the generator "ϕ" and a consecutive generator of the same system are proportional to

$$\frac{1}{a}\sin\phi,\quad -\frac{1}{b}\cos\phi,\quad \frac{1}{c}.$$

The coordinates of any point, O, on the generator are

$$a(\cos\phi-k\sin\phi),\quad b(\sin\phi+k\cos\phi),\quad ck.$$

If O is the central point the normal at O is perpendicular to the S.D. between the generator and a consecutive generator of the same system. Hence we find

$$k=\frac{c^2(b^2-a^2)\sin\phi\cos\phi}{a^2b^2+b^2c^2\sin^2\phi+c^2a^2\cos^2\phi},$$

and the coordinates of the central point are given by

$$\frac{x}{a^3(b^2+c^2)\cos\phi}=\frac{y}{b^3(c^2+a^2)\sin\phi}=\frac{z}{c^3(b^2-a^2)\sin\phi\cos\phi}$$
$$=\frac{1}{a^2b^2+b^2c^2\sin^2\phi+c^2a^2\cos^2\phi}.$$

Ex. 6. Find the locus of the central points of the generators of the hyperboloid.

The equation to a surface containing the central points is obtained by eliminating ϕ between the equations for the coordinates. It is

$$\frac{a^6}{x^2}(b^2+c^2)^2+\frac{b^6}{y^2}(c^2+a^2)^2-\frac{c^6}{z^2}(b^2-a^2)^2=0.$$

Ex. 7. For the generator of the paraboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$ given by $\frac{x}{a}-\frac{y}{b}=2\lambda$, $\frac{x}{a}+\frac{y}{b}=\frac{z}{\lambda}$, prove that the parameter of distribution is $\frac{ab(a^2+b^2+4\lambda^2)}{a^2+b^2}$, and that the central point is

$$\left(\frac{2a^3\lambda}{a^2+b^2}, \frac{-2b^3\lambda}{a^2+b^2}, \frac{2(a^2-b^2)\lambda^2}{a^2+b^2}\right).$$

Prove also that the central points of the systems of generators lie on the planes $\frac{x}{a^3}\pm\frac{y}{b^3}=0$.

Ex. 8. If G is a given generator of a hyperboloid, prove that the tangent plane at the central point of G is perpendicular to the tangent plane to the asymptotic cone whose generator of contact is parallel to G.

Ex. 9. A pair of planes through a given generator of a hyperboloid touch the surface at points A and B, and contain the normals at points A′ and B′ of the generator. If θ is the angle between them, prove that $\tan^2\theta=-\frac{AB\,.\,A'B'}{AB'\,.\,A'B}$.

Ex. 10. If the tangent plane at a point P of a generator, central point O, makes an angle θ with the tangent plane at O, $p\tan\theta=OP$, where p is the parameter of distribution.

*Examples VI.

1. Prove that the line $lx+my+nz+p=0$, $l'x+m'y+n'z+p'=0$ is a generator of the hyperboloid $x^2/a+y^2/b+z^2/c=1$ if $al^2+bm^2+cn^2=p^2$, $al'^2+bm'^2+cn'^2=p'^2$, and $all'+bmm'+cnn'=pp'$.

2. Shew that the equations

$$y-\lambda z+\lambda+1=0, \quad (\lambda+1)x+y+\lambda=0$$

represent for different values of λ generators of one system of the hyperboloid $yz+zx+xy+1=0$, and find the equations to generators of the other system.

3. Tangent planes to $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, which are parallel to tangent planes to

$$\frac{b^2c^2x^2}{c^2-b^2}+\frac{c^2a^2y^2}{c^2-a^2}+\frac{a^2b^2z^2}{a^2+b^2}=0,$$

cut the surface in perpendicular generators.

4. The shortest distances between generators of the same system drawn at the ends of diameters of the principal elliptic section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ lie on the surfaces whose equations are

$$\frac{cxy}{x^2+y^2}=\pm\frac{abz}{a^2-b^2}.$$

5. Shew that the shortest distance of any two perpendicular members of that system of generators of the paraboloid $y(ax+by)=z$, which is perpendicular to the y-axis, lies in the plane $a^2z=b$.

6. Prove that any point on the lines

$$x+1=\mu y=-(\mu+1)z$$

lies on the surface $$yz+zx+xy+y+z=0,$$

and find equations to determine the other system of lines which lies on the surface.

7. The four conicoids, each of which passes through three of four given non-intersecting lines, have two common generators.

8. Prove that the equation to the conicoid through the lines

$$u=0=v, \quad u'=0=v',$$

$$\lambda u+\mu v+\lambda' u'+\mu' v'=0=lu+mv+l'u'+m'v'$$

is $$\frac{\lambda u+\mu v}{\lambda' u'+\mu' v'}=\frac{lu+mv}{l'u'+m'v'}.$$

9. **ABC**, **A′B′C′** are two given triangles. **P** moves so that the lines through **P** which meet the pairs of corresponding sides **AB**, **A′B′**; **BC**, **B′C′**; **CA**, **C′A′** are coplanar. Prove that the locus of **P** is the hyperboloid through **AA′**, **BB′**, and **CC′**.

10. If from a fixed point on a hyperboloid lines are drawn to intersect the diagonals of the quadrilaterals formed by two fixed and two variable generators, these lines are coplanar.

11. Through a variable generator

$$x-y=\lambda, \quad x+y=2z/\lambda$$

of the paraboloid $x^2-y^2=2z$ a plane is drawn making a constant angle α with the plane $x=y$. Find the locus of the point at which it touches the paraboloid.

12. Prove that the locus of the line of intersection of two perpendicular planes which pass through two fixed non-intersecting lines is a hyperboloid whose central circular sections are perpendicular to the lines and have their diameters equal to their shortest distance.

13. Prove that if the generators of $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ be drawn through the points where it is met by a tangent to

$$z=0, \quad \frac{x^2}{a^2(a^2+c^2)}+\frac{y^2}{b^2(b^2+c^2)}=\frac{1}{a^2+b^2},$$

they form a skew quadrilateral with two opposite angles right angles, and the other diagonal of which is a generator of the cylinder

$$\frac{x^2(a^2+c^2)}{a^2}+\frac{y^2(b^2+c^2)}{b^2}=a^2+b^2.$$

14. The normals to $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ at points of a generator meet the plane $z=0$ at points lying on a straight line, and for different generators of the same system this line touches a fixed conic.

15. Prove that the generators of $ax^2+by^2+cz^2=1$ through (x_1, y_1, z_1), (x_2, y_2, z_2) lie in the planes

$$(ax_1x_2+by_1y_2+cz_1z_2-1)(ax^2+by^2+cz^2-1)$$
$$=2(axx_1+byy_1+czz_1-1)(axx_2+byy_2+czz_2-1).$$

16. The generators through points on the principal elliptic section of $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, such that the eccentric angle of the one is double the eccentric angle of the other, intersect on the curves given by

$$x=\frac{a(1-3t^2)}{1+t^2},\quad y=\frac{bt(3-t^2)}{1+t^2},\quad z=\pm ct.$$

17. The planes of triangles which have a fixed centre of gravity and have their vertices on three given straight lines which are parallel to the same plane, touch a cone of the second degree, and their sides are generators of three paraboloids.

18. The cubic curve

$$x=\frac{1}{\lambda-\alpha},\quad y-\frac{1}{\lambda-\beta},\quad z=\frac{1}{\lambda-\gamma}$$

meets the conicoid $ax^2+by^2+cz^2=1$ in six points, and the normals at these points are generators of the hyperboloid

$$ayz(\beta-\gamma)+bzx(\gamma-\alpha)+cxy(\alpha-\beta)+x(b-c)+y(c-a)+z(a-b)=0.$$

19. Prove that the locus of a point whose distances from two given lines are in a constant ratio is a hyperboloid of one sheet, and that the projections of the lines on the tangent plane at the point where it meets the shortest distance form a harmonic pencil with the generators through the point.

20. The generators through **P** on the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ meet the plane $z=0$ in **A** and **B**. If **PA** : **PB** is constant, find the locus of **P**.

21. If the median of the triangle **PAB** in the last example is parallel to the fixed plane $\alpha x+\beta y+\gamma z=0$, shew that **P** lies on the surface

$$z(\alpha x+\beta y)+\gamma(c^2+z^2)=0.$$

22. If **A** and **B** are the extremities of conjugate diameters of the principal elliptic section, prove that the median through **P** of the triangle **PAB** lies on the cone

$$\frac{2x^2}{a^2}+\frac{2y^2}{b^2}=\left(\frac{z}{c}+1\right)^2.$$

23. **A** and **B** are the extremities of the axes of the principal elliptic section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, and **T** is any line in the plane of the section. $\mathbf{G}_1$, $\mathbf{G}_2$ are generators of the same system, $\mathbf{G}_1$ passing through **A** and $\mathbf{G}_2$ through **B**. Two hyperboloids are drawn, one through **T**, $\mathbf{G}_1$, **OZ**, the other through **T**, $\mathbf{G}_2$, **OZ**. Shew that he other common generators of these hyperboloids lie on the surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}\pm\frac{z}{c}\left(\frac{x}{a}-\frac{y}{b}\right)-\frac{x}{a}-\frac{y}{b}=0.$$

24. Prove that the shortest distances between the generator

$$\frac{x}{a}=\frac{z}{c}\cos\alpha-\sin\alpha,\quad \frac{y}{b}=\frac{z}{c}\sin\alpha+\cos\alpha,$$

and the other generators of the same system, meet the generators in points lying in the plane

$$\frac{x\cos\alpha}{a}(a^{-2}-b^{-2}+c^{-2})+\frac{y\sin\alpha}{b}(-a^{-2}+b^{-2}+c^{-2})+\frac{z}{c}(a^{-2}+b^{-2}+c^{-2})=0.$$

25. If the generators through P, a point on the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

whose centre is O, meet the plane $z=0$ in A and B, and the volume of the tetrahedron OAPB is constant and equal to $abc/6$, P lies on one of the planes $z=\pm c$.

CHAPTER X.

CONFOCAL CONICOIDS.

115. Confocal conicoids are conicoids whose principal sections have the same foci. Thus the equation

$$\frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}+\frac{z^2}{c^2-\lambda}=1$$

represents, for any value of λ, a conicoid confocal with

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

since the sections of the conicoids by the planes **YOZ**, **ZOX**, **XOY** are confocal conics. Again, if arbitrary values are assigned to α in the equation

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\alpha^2-b^2}+\frac{z^2}{\alpha^2-c^2}=1,$$

b and c being constants, we obtain the equations to a system of confocal conicoids. If this form of equation be chosen to represent a confocal α is called the **primary semi-axis.**

The sections of the paraboloids

$$\frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}=2z-\lambda,\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=2z,$$

by the planes **YOZ**, **ZOX**, consist of confocal parabolas, and hence the paraboloids are confocal.

116. The three confocals through a point. *Through any point there pass three conicoids confocal with a given ellipsoid,—an ellipsoid, a hyperboloid of one sheet, and a hyperboloid of two sheets.*

The equation $\dfrac{x^2}{a^2-\lambda}+\dfrac{y^2}{b^2-\lambda}+\dfrac{z^2}{c^2-\lambda}=1$ represents any conicoid confocal with the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$. If the confocal passes through (α, β, γ),

$$\frac{\alpha^2}{a^2-\lambda}+\frac{\beta^2}{b^2-\lambda}+\frac{\gamma^2}{c^2-\lambda}=1,$$

or $$f(\lambda)\equiv(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)-\alpha^2(b^2-\lambda)(c^2-\lambda)$$
$$-\beta^2(c^2-\lambda)(a^2-\lambda)-\gamma^2(a^2-\lambda)(b^2-\lambda)=0.$$

This cubic equation in λ gives the parameters of three confocals which pass through (α, β, γ). Suppose that $a>b>c$. When

$\lambda=$	∞,	a^2,	b^2,	c^2,	$-\infty$,
$f(\lambda)$ is	$-$,	$-$,	$+$,	$-$,	$+$.

Hence the equation $f(\lambda)=0$ has three real roots $\lambda_1, \lambda_2, \lambda_3$ such that $$a^2>\lambda_1>b^2>\lambda_2>c^2>\lambda_3.$$

Therefore the confocal is a hyperboloid of two sheets, a hyperboloid of one sheet, or an ellipsoid, according as $\lambda=\lambda_1, \lambda_2$, or λ_3.

As λ tends to c^2 the confocal ellipsoid tends to coincide with that part of the plane XOY enclosed within the ellipse $z=0$, $\dfrac{x^2}{a^2-c^2}+\dfrac{y^2}{b^2-c^2}=1$; and the confocal hyperboloid of one sheet tends to coincide with that part of the plane which lies without the ellipse. As λ tends to b^2 the confocal hyperboloid of one sheet tends to coincide with that part of the plane ZOX which lies between the two branches of the hyperbola $y=0$, $\dfrac{x^2}{a^2-b^2}+\dfrac{z^2}{c^2-b^2}=1$; and the confocal hyperboloid of two sheets tends to coincide with the two portions of the plane which are enclosed by the two branches of the hyperbola. If $\lambda=a^2$, the confocal is imaginary. The above ellipse and hyperbola are called the **focal conics.**

Ex. 1. Three paraboloids confocal with a given paraboloid pass through a given point,—two elliptic and one hyperbolic.

Ex. 2. Prove that the equation to the confocal through the point of the focal ellipse whose eccentric angle is α is

$$\frac{x^2}{(a^2-b^2)\cos^2\alpha}-\frac{y^2}{(a^2-b^2)\sin^2\alpha}+\frac{z^2}{c^2-a^2\sin^2\alpha-b^2\cos^2\alpha}=1.$$

Ex. 3. Prove that the equation to the confocal which has a system of circular sections parallel to the plane $x=y$ is

$$\frac{x^2}{(c^2-a^2)(a^2-b^2)}+\frac{y^2}{(b^2-c^2)(a^2-b^2)}-\frac{z^2}{2(b^2-c^2)(c^2-a^2)}=\frac{1}{2c^2-a^2-b^2}.$$

117. Elliptic coordinates. Since $\lambda_1, \lambda_2, \lambda_3$ are the roots of the equation $f(\lambda)=0$,

$$f(\lambda)\equiv-(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3).$$

Therefore

$$1-\frac{\alpha^2}{a^2-\lambda}-\frac{\beta^2}{b^2-\lambda}-\frac{\gamma^2}{c^2-\lambda}\equiv\frac{f(\lambda)}{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}$$
$$\equiv\frac{-(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)}{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}.$$

Hence, by the rule for partial fractions,

$$\alpha^2=\frac{(a^2-\lambda_1)(a^2-\lambda_2)(a^2-\lambda_3)}{(b^2-a^2)(c^2-a^2)},\quad \beta^2=\frac{(b^2-\lambda_1)(b^2-\lambda_2)(b^2-\lambda_3)}{(c^2-b^2)(a^2-b^2)},$$
$$\gamma^2=\frac{(c^2-\lambda_1)(c^2-\lambda_2)(c^2-\lambda_3)}{(a^2-c^2)(b^2-c^2)}.$$

These express the coordinates α, β, γ of a point P, in terms of the parameters of the confocals of a given conicoid that pass through P; and if the parameters are given, and the octant in which P lies is known, the position of P is uniquely determined. Hence λ_1, λ_2, λ_3 are called the **elliptic coordinates** of P with reference to the fundamental conicoid $x^2/a^2+y^2/b^2+z^2/c^2=1$.

Ex. 1. If a_1, a_2, a_3 are the primary semi-axes of the confocals to $x^2/a^2+y^2/b^2+z^2/c^2=1$ which pass through a point (α, β, γ),

$$\alpha^2=\frac{a_1^2a_2^2a_3^2}{(b^2-a^2)(c^2-a^2)},$$
$$\beta^2=\frac{(b^2-a^2+a_1^2)(b^2-a^2+a_2^2)(b^2-a^2+a_3^2)}{(c^2-b^2)(a^2-b^2)},$$
$$\gamma^2=\frac{(c^2-a^2+a_1^2)(c^2-a^2+a_2^2)(c^2-a^2+a_3^2)}{(a^2-c^2)(b^2-c^2)}.$$

Ex. 2. What loci are represented by the equations in elliptic coordinates,

(i) $\lambda_1+\lambda_2+\lambda_3=\text{constant}$,

(ii) $\lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2=\text{constant}$,

(iii) $\lambda_1\lambda_2\lambda_3=\text{constant}$?

Ex. 3. If λ_1, λ_2, λ_3 are the parameters of the paraboloids confocal to $\frac{x^2}{a}+\frac{y^2}{b}=2z$ which pass through the point (α, β, γ), prove that

$$\alpha^2=\frac{(a-\lambda_1)(a-\lambda_2)(a-\lambda_3)}{b-a}, \quad \beta^2=\frac{(b-\lambda_1)(b-\lambda_2)(b-\lambda_3)}{a-b},$$

$$\gamma=\frac{\lambda_1+\lambda_2+\lambda_3-a-b}{2}.$$

118. Confocals cut at right angles. *The tangent planes to two confocals at any common point are at right angles.*

Let (x_1, y_1, z_1) be a point common to the confocals to $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, whose parameters are λ_1 and λ_2.

Then
$$\frac{x_1^2}{a^2-\lambda_1}+\frac{y_1^2}{b^2-\lambda_1}+\frac{z_1^2}{c^2-\lambda_1}=1$$

and
$$\frac{x_1^2}{a^2-\lambda_2}+\frac{y_1^2}{b^2-\lambda_2}+\frac{z_1^2}{c^2-\lambda_2}=1.$$

Therefore, subtracting,

$$\frac{x_1^2}{(a^2-\lambda_1)(a^2-\lambda_2)}+\frac{y_1^2}{(b^2-\lambda_1)(b^2-\lambda_2)}+\frac{z_1^2}{(c^2-\lambda_1)(c^2-\lambda_2)}=0,$$

and this is the condition that the tangent planes at (x_1, y_1, z_1) to the confocals should be at right angles.

Cor. The tangent planes at a point to the three confocals which pass through it are mutually perpendicular.

119. Confocal touching given plane. *One conicoid confocal with a given conicoid touches a given plane.*

For the condition that the plane $lx+my+nz=p$ should touch the conicoid

$$\frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}+\frac{z^2}{c^2-\lambda}=1,$$

viz., $$p^2=(a^2-\lambda)l^2+(b^2-\lambda)m^2+(c^2-\lambda)n^2,$$

determines one value of λ.

Ex. 1. A given plane and the parallel tangent plane to a conicoid are at distances p and p_0 from the centre. Prove that the parameter of the confocal conicoid which touches the plane is $p_0^2-p^2$.

Ex. 2. Prove that the perpendiculars from the origin to the tangent planes to the ellipsoid which touch it along its curve of intersection with the confocal whose parameter is λ lie on the cone

$$\frac{a^2x^2}{a^2-\lambda}+\frac{b^2y^2}{b^2-\lambda}+\frac{c^2z^2}{c^2-\lambda}=0.$$

120. Confocals touching given line. *Two conicoids confocal with a given conicoid touch a given line and the tangent planes at the points of contact are at right angles.*

The condition that the line $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$ should touch the conicoid

$$\frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}+\frac{z^2}{c^2-\lambda}=1,$$

viz.,
$$\left(\frac{l^2}{a^2-\lambda}+\frac{m^2}{b^2-\lambda}+\frac{n^2}{c^2-\lambda}\right)\left(\frac{\alpha^2}{a^2-\lambda}+\frac{\beta^2}{b^2-\lambda}+\frac{\gamma^2}{c^2-\lambda}-1\right)$$
$$=\left(\frac{\alpha l}{a^2-\lambda}+\frac{\beta m}{b^2-\lambda}+\frac{\gamma n}{c^2-\lambda}\right)^2,$$

or,
$$\Sigma\frac{(\beta n-\gamma m)^2}{(b^2-\lambda)(c^2-\lambda)}=\frac{l^2}{a^2-\lambda}+\frac{m^2}{b^2-\lambda}+\frac{n^2}{c^2-\lambda},$$

gives two values of λ.

Let the equations to the two confocals be

$$\frac{x^2}{a^2-\lambda_1}+\frac{y^2}{b^2-\lambda_1}+\frac{z^2}{c^2-\lambda_1}=1, \quad\text{.................(1)}$$
$$\frac{x^2}{a^2-\lambda_2}+\frac{y^2}{b^2-\lambda_2}+\frac{z^2}{c^2-\lambda_2}=1, \quad\text{.................(2)}$$

and let the line touch the first at P, (x_1, y_1, z_1) and the second at Q, (x_2, y_2, z_2). Then, since PQ lies in the tangent planes to the confocals at P and Q,

$$\frac{x_1x_2}{a^2-\lambda_1}+\frac{y_1y_2}{b^2-\lambda_1}+\frac{z_1z_2}{c^2-\lambda_1}=1 \quad\text{and}\quad \frac{x_1x_2}{a^2-\lambda_2}+\frac{y_1y_2}{b^2-\lambda_2}+\frac{z_1z_2}{c^2-\lambda_2}=1.$$

Therefore, subtracting,

$$\frac{x_1x_2}{(a^2-\lambda_1)(a^2-\lambda_2)}+\frac{y_1y_2}{(b^2-\lambda_1)(b^2-\lambda_2)}+\frac{z_1z_2}{(c^2-\lambda_1)(c^2-\lambda_2)}=0,$$

which is the condition that the tangent planes should be at right angles.

121. Parameters of confocals through a point on a conicoid. *If* P *is a point on a central conicoid, the parameters of the two confocals of the conicoid which pass through* P *are equal to the squares of the semi-axes of the central section of the conicoid which is parallel to the tangent plane at* P, *and the normals to the confocals at* P *are parallel to the axes.*

Let P, (x_1, y_1, z_1) lie on the conicoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Then the parameters of the confocals through P are given by the equation

$$\frac{x_1^2}{a^2-\lambda}+\frac{y_1^2}{b^2-\lambda}+\frac{z_1^2}{c^2-\lambda}=1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2},$$

or,
$$\frac{x_1^2}{a^2(a^2-\lambda)}+\frac{y_1^2}{b^2(b^2-\lambda)}+\frac{z_1^2}{c^2(c^2-\lambda)}=0.$$

But the squares of the semi-axes of the section of the conicoid by the plane $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=0$ are given by, (§ 86),

$$\frac{x_1^2}{a^2(a^2-r^2)}+\frac{y_1^2}{b^2(b^2-r^2)}+\frac{z_1^2}{c^2(c^2-r^2)}=0.$$

Therefore the values of λ are the values of r^2. Again, the direction-cosines of the semi-axis of length r are given by

$$\frac{l}{\dfrac{x_1}{a^2-r^2}}=\frac{m}{\dfrac{y_1}{b^2-r^2}}=\frac{n}{\dfrac{z_1}{c^2-r^2}},$$

and therefore the axis is parallel to the normal at (x_1, y_1, z_1) to the confocal whose parameter is equal to r^2.

122. Locus of poles of plane with respect to confocals. *The locus of the poles of a given plane with respect to the conicoids confocal with a given conicoid is the normal to the plane at the point of contact with that confocal which touches it.*

Let a confocal be represented by

$$\frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}+\frac{z^2}{c^2-\lambda}=1,$$

and the given plane by $lx+my+nz=1$.

Then, if (ξ, η, ζ) is the pole of the plane with respect to the confocal,

$$l=\frac{\xi}{a^2-\lambda}, \quad m=\frac{\eta}{b^2-\lambda}, \quad n=\frac{\zeta}{c^2-\lambda}.$$

Whence $$\frac{\xi}{l}-a^2=\frac{\eta}{m}-b^2=\frac{\zeta}{n}-c^2.$$

Therefore the locus of (ξ, η, ζ) is a straight line at right angles to the given plane. Again, the pole of the plane with respect to that confocal which touches it is the point of contact. Hence the point of contact is on the locus, which is therefore the normal to the plane at the point of contact.

123. Normals to the three confocals through a point. *Three conicoids confocal with a given conicoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

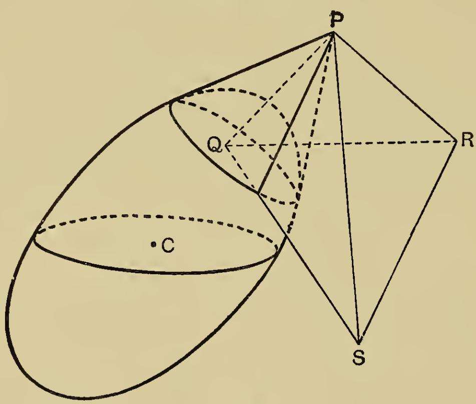

Fig. 47.

pass through a given point P, *and* PQ, PR, PS, *the normals at* P *to the confocals, meet the polar plane of* P *with respect to the given conicoid in* Q, R, S. *To prove that*

$$PQ=\lambda_1/p_1, \quad PR=\lambda_2/p_2, \quad PS=\lambda_3/p_3,$$

where p_1, p_2, p_3 *are the perpendiculars from the centre to the tangent planes at* P *to the confocals, and* $\lambda_1, \lambda_2, \lambda_3$ *are the parameters of the confocals.*

If the coordinates of P, (fig. 47), are (α, β, γ), the equations to PQ are

$$\frac{x-\alpha}{\dfrac{p_1\alpha}{a^2-\lambda_1}}=\frac{y-\beta}{\dfrac{p_1\beta}{b^2-\lambda_1}}=\frac{z-\gamma}{\dfrac{p_1\gamma}{c^2-\lambda_1}} \quad (=r).$$

Hence, if $PQ=r$, the coordinates of Q are

$$\alpha\left(1+\frac{p_1 r}{a^2-\lambda_1}\right),\quad \beta\left(1+\frac{p_1 r}{b^2-\lambda_1}\right),\quad \gamma\left(1+\frac{p_1 r}{c^2-\lambda_1}\right).$$

But Q is on the polar plane of P, and therefore

$$\frac{\alpha^2}{a^2}\left(1+\frac{p_1 r}{a^2-\lambda_1}\right)+\frac{\beta^2}{b^2}\left(1+\frac{p_1 r}{b^2-\lambda_1}\right)+\frac{\gamma^2}{c^2}\left(1+\frac{p_1 r}{c^2-\lambda_1}\right)=1$$

$$=\frac{\alpha^2}{a^2-\lambda_1}+\frac{\beta^2}{b^2-\lambda_1}+\frac{\gamma^2}{c^2-\lambda_1}.$$

Rearranging, this becomes

$$(p_1 r-\lambda_1)\left\{\frac{\alpha^2}{a^2(a^2-\lambda_1)}+\frac{\beta^2}{b^2(b^2-\lambda_1)}+\frac{\gamma^2}{c^2(c^2-\lambda_1)}\right\}=0.$$

Therefore $r=\lambda_1/p_1$. Similarly, $PR=\lambda_2/p_2$ and $PS=\lambda_3/p_3$.

124. *The tetrahedron* PQRS *is self-polar with respect to the given conicoid.*

Substituting λ_1 for $p_1 r$, the coordinates of Q become $\dfrac{a^2\alpha}{a^2-\lambda_1}, \dfrac{b^2\beta}{b^2-\lambda_1}, \dfrac{c^2\gamma}{c^2-\lambda_1}$. Whence the polar plane of Q with respect to the conicoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1$$

is given by
$$\frac{\alpha x}{a^2-\lambda_1}+\frac{\beta y}{b^2-\lambda_1}+\frac{\gamma z}{c^2-\lambda_1}=1,$$

and therefore is the tangent plane at P to the confocal whose parameter is λ_1, or is the plane PRS. Similarly, the polar planes of R and S are the planes PQS, PQR, and, by hypothesis, the polar plane of P is the plane QRS.

125. Axes of enveloping cone. *The normals to the three confocals through* P *are the axes of the enveloping cone whose vertex is* P.

Since the tetrahedron PQRS is self-polar with respect to the conicoid, the triangle QRS is self-polar with respect

to the common section of the conicoid and enveloping cone by the plane QRS. Therefore, (§ 78), PQ, PR, PS are conjugate diameters of the cone, and being mutually perpendicular, are the principal axes.

126. Equation to enveloping cone. *To find the equation to the enveloping cone whose vertex is* P *referred to its principal axes.*

The equation will be of the form $Ax^2+By^2+Cz^2=0$. Since the tangent planes at P to the confocals are the coordinate planes, C, the centre of the given conicoid, is (p_1, p_2, p_3), and the equations to PC are $x/p_1=y/p_2=z/p_3$. But the centre of the section of the cone or conicoid by the plane QRS lies on PC, and therefore its coordinates are of the form kp_1, kp_2, kp_3, and the equation to the plane QRS is, (§ 71),

$$(x-kp_1)Ap_1+(y-kp_2)Bp_2+(z-kp_3)Cp_3=0.$$

By § 123, the plane QRS makes intercepts λ_1/p_1, λ_2/p_2, λ_3/p_3 on the axes, and therefore its equation is also

$$\frac{p_1x}{\lambda_1}+\frac{p_2y}{\lambda_2}+\frac{p_3z}{\lambda_3}=1.$$

Therefore $$\frac{A}{1/\lambda_1}=\frac{B}{1/\lambda_2}=\frac{C}{1/\lambda_3},$$

and the equation to the cone is

$$\frac{x^2}{\lambda_1}+\frac{y^2}{\lambda_2}+\frac{z^2}{\lambda_3}=0.$$

127. Equation to conicoid. *To find the equation to the given conicoid referred to the normals to the confocals through* P *as coordinate axes.*

The equation will be of the form

$$\frac{x^2}{\lambda_1}+\frac{y^2}{\lambda_2}+\frac{z^2}{\lambda_3}=k\left(\frac{p_1x}{\lambda_1}+\frac{p_2y}{\lambda_2}+\frac{p_3z}{\lambda_3}-1\right)^2. \quad \text{..........(1)}$$

The centre C, (p_1, p_2, p_3) bisects all chords through it. The equations to the chord parallel to OX are

$$\frac{x-p_1}{1}=\frac{y-p_2}{0}=\frac{z-p_3}{0} \quad (=r),$$

and hence the equation obtained by substituting p_1+r, p_2, p_3 for x, y, z, in (1), viz.,

$$\frac{(p_1+r)^2}{\lambda_1}+\frac{p_2^2}{\lambda_2}+\frac{p_3^2}{\lambda_3}=k\left\{\frac{p_1(p_1+r)}{\lambda_1}+\frac{p_2^2}{\lambda_2}+\frac{p_3^2}{\lambda_3}-1\right\}^2,$$

takes the form $\mathsf{L}r^2+\mathsf{M}=0$. Therefore

$$\frac{1}{k}=\frac{p_1^2}{\lambda_1}+\frac{p_2^2}{\lambda_2}+\frac{p_3^2}{\lambda_3}-1,$$

and the equation to the conicoid is

$$\left(\frac{x^2}{\lambda_1}+\frac{y^2}{\lambda_2}+\frac{z^2}{\lambda_3}\right)\left(\frac{p_1^2}{\lambda_1}+\frac{p_2^2}{\lambda_2}+\frac{p_3^2}{\lambda_3}-1\right)=\left(\frac{p_1x}{\lambda_1}+\frac{p_2y}{\lambda_2}+\frac{p_3z}{\lambda_3}-1\right)^2.$$

Ex. 1. If λ and μ are the parameters of the confocal hyperboloids through a point P on the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1,$$

prove that the perpendicular from the centre to the tangent plane at P to the ellipsoid is $\dfrac{abc}{\sqrt{\lambda\mu}}$. Prove also that the perpendiculars to the tangent planes to the hyperboloids are

$$\sqrt{\frac{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}{\lambda(\lambda-\mu)}},\quad \sqrt{\frac{(a^2-\mu)(b^2-\mu)(c^2-\mu)}{\mu(\mu-\lambda)}}.$$

Ex. 2. If λ_1, λ_2, λ_3 are the parameters of the three confocals to

$$x^2/a^2+y^2/b^2+z^2/c^2=1$$

that pass through P, prove that the perpendiculars from the centre to the tangent plane at P are

$$\sqrt{\frac{(a^2-\lambda_1)(b^2-\lambda_1)(c^2-\lambda_1)}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}},\quad \text{etc.}$$

Ex. 3. If a_1, b_1, c_1; a_2, b_2, c_2; a_3, b_3, c_3 are the axes of the confocals to

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1$$

which pass through a point (x, y, z), and p_1, p_2, p_3 are the perpendiculars from the centre to the tangent planes to the confocals at the point, prove that

$$x^2+y^2+z^2=a_1^2+b_2^2+c_3^2,\quad \frac{p_1^2}{a_1^2}+\frac{p_2^2}{a_2^2}+\frac{p_3^2}{a_3^2}=1,$$

$$\frac{p_1^2}{a_1^2-\alpha^2}+\frac{p_2^2}{a_2^2-\alpha^2}+\frac{p_3^2}{a_3^2-\alpha^2}-1=\frac{\alpha^2\beta^2\gamma^2}{(a_1^2-\alpha^2)(a_2^2-\alpha^2)(a_3^2-\alpha^2)}.$$

Ex. 4. If a_1, b_1, c_1; a_2, b_2, c_2; a_3, b_3, c_3 are the axes of the confocals to a given conicoid through P, show that the equations, referred to the normals at P to the confocals, of the cones with P as vertex and the focal conics as bases, are

$$\frac{x^2}{b_1^2}+\frac{y^2}{b_2^2}+\frac{z^2}{b_3^2}=0,\quad \frac{x^2}{c_1^2}+\frac{y^2}{c_2^2}+\frac{z^2}{c_3^2}=0.$$

Ex. 5. Prove that the direction-cosines of the four common generators of the cones satisfy the equations

$$\frac{l^2}{p_1^2/a_1^2}=\frac{m^2}{p_2^2/a_2^2}=\frac{n^2}{p_3^2/a_3^2}.$$

(The intercepts on these generators by the ellipsoid are called the **bifocal chords** of the ellipsoid through the point P.)

Ex. 6. Prove that the bifocal chords of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

through a point P on the surface lie on a right circular cone whose axis is the normal at P and who sesemi-vertical angle is $\cos^{-1}\frac{bc}{\sqrt{\lambda_1\lambda_2}}$, where λ_1, λ_2 are the parameters of the confocals through P.

Ex. 7. If the plane through the centre parallel to the tangent plane at P meets one of the bifocal chords through P in F, then $PF=a$.

Ex. 8. P is any point on the curve of intersection of an ellipsoid and a given confocal and r is the length of the central radius of the ellipsoid which is parallel to the tangent to the curve at P. If p is the perpendicular from the centre to the tangent plane to the ellipsoid at P, prove that pr is constant.

CORRESPONDING POINTS.

128. Two points, P, (x, y, z) and Q, (ξ, η, ζ), situated respectively on the conicoids

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \quad \frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1$$

are said to correspond when

$$\frac{x}{a}=\frac{\xi}{\alpha}, \quad \frac{y}{b}=\frac{\eta}{\beta}, \quad \frac{z}{c}=\frac{\gamma}{\zeta}.$$

If P *and* Q *are any points on an ellipsoid and* P′ *and* Q′ *are the corresponding points on a confocal ellipsoid,* PQ′ = P′Q.

Let P and Q, (x, y, z), (ξ, η, ζ) lie on the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

and let P′ and Q′, (x', y', z'), (ξ', η', ζ') be the corresponding points on the confocal

$$\frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}+\frac{z^2}{c^2-\lambda}=1.$$

Then $$\frac{x}{a}=\frac{x'}{\sqrt{a^2-\lambda}},\quad \frac{\xi}{a}=\frac{\xi'}{\sqrt{a^2-\lambda}},\ \text{etc.}$$

Therefore

$$(x-\xi')^2-(x'-\xi)^2=\left(x-\frac{\sqrt{a^2-\lambda}}{a}\xi\right)^2-\left(\frac{\sqrt{a^2-\lambda}}{a}x-\xi\right)^2$$

$$=\lambda\left(\frac{x^2}{a^2}-\frac{\xi^2}{a^2}\right),$$

and hence

$$PQ'^2-P'Q^2=\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}-\frac{\zeta^2}{c^2}\right)=0,$$

which proves the proposition.

Ex. 1. If P is a point on an ellipsoid and P′ is the corresponding point on a confocal whose parameter is λ, $OP^2-OP'^2=\lambda$, where O is the centre.

Ex. 2. OP, OQ, OR are conjugate diameters of an ellipsoid, and P′, Q′, R′ are the points of a concentric sphere corresponding to P, Q, R. Prove that OP′, OQ′, OR′ are mutually perpendicular.

Ex. 3. If P″, Q″, R″ are the corresponding points on a coaxal ellipsoid, OP″, OQ″, OR″ are conjugate diameters.

Ex. 4. An umbilic on an ellipsoid corresponds to an umbilic on any confocal ellipsoid.

Ex. 5. P and Q are any points on a generator of a hyperboloid and P′ and Q′ are the corresponding points on a second hyperboloid. Prove that P′ and Q′ lie on a generator, and that PQ = P′Q′.

THE FOCI OF CONICOIDS.

***129.** [1](I) *The locus of a point such that the square on its distance from a given point is in a constant ratio to the rectangle contained by its distances from two fixed planes is a conicoid.*

The equation to the locus is of the form

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2$$
$$=k^2(lx+my+nz+p)(l'x+m'y+n'z+p'),$$

which represents a conicoid.

[1](II) *The locus of a point whose distance from a fixed point is in a constant ratio to its distance, measured parallel to a given plane, from a given line, is a conicoid.*

---

[1] That a conicoid could be generated by the method (I) was first pointed out by Salmon. The method (II) is due to MacCullagh.

Choose rectangular axes so that the given plane is the xy-plane and the point of intersection of the given line and given plane is the origin. Let the fixed point be (α, β, γ) and the fixed line have equations $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$. The plane through (ξ, η, ζ) parallel to the xy-plane meets the given line in the point $\left(\frac{l\zeta}{n}, \frac{m\zeta}{n}, \zeta\right)$, and therefore the distance of (ξ, η, ζ) from the line measured parallel to the given plane is given by

$$\left\{\left(\xi-\frac{l\zeta}{n}\right)^2+\left(\eta-\frac{m\zeta}{n}\right)^2\right\}^{\frac{1}{2}}.$$

Hence the equation to the locus is

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=k^2\left\{\left(x-\frac{lz}{n}\right)^2+\left(y-\frac{mz}{n}\right)^2\right\},$$

which represents a conicoid.

In (I) the equation to the locus is of the form $\lambda\phi-uv=0$, and in (II) of the form $\lambda\phi-(u^2+v^2)=0$, where

$$\phi\equiv(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2,$$

and $u=0$, $v=0$ represent planes. In either case, if $S=0$ is the equation to the locus, the equation $S-\lambda\phi=0$ represents a pair of planes. In (I) the planes are real, in (II) they are imaginary, but the line of intersection, $u=0$, $v=0$, is real in both cases. These suggest the following definition of the foci and directrices of a conicoid:

If $S=0$ *is the equation to a conicoid and* $\lambda, \alpha, \beta, \gamma$ *can be found so that the equation* $S-\lambda\phi=0$ *represents two planes, real or imaginary,* (α, β, γ) *is a* **focus**, *and the line of intersection of the planes is the corresponding* **directrix.**

If the planes are real we shall call (α, β, γ) a focus of the first species, if they are imaginary, a focus of the second species.

Lemma. *If the equation* $F(x, y, z)=0$ *represents a pair of planes, the equations* $\frac{\partial F}{\partial x}=0$, $\frac{\partial F}{\partial y}=0$, $\frac{\partial F}{\partial z}=0$ *represent three planes passing through their line of intersection.*

If $\mathsf{F}(x, y, z)=uv$, where $u=ax+by+cz+d$ and $v=a'x+b'y+c'z+d'$,

then $$\frac{\partial \mathsf{F}}{\partial x}=av+a'u, \quad \frac{\partial \mathsf{F}}{\partial y}=bv+b'u, \quad \frac{\partial \mathsf{F}}{\partial z}=cv+c'u,$$

whence the proposition is evident.

***130. Foci of ellipsoid and paraboloids.** *To find the foci of the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \quad (a>b>c).$$

The equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1-\lambda(x-\alpha)^2-\lambda(y-\beta)^2-\lambda(z-\gamma)^2=0 \quad \ldots(1)$$

is to represent a pair of planes, and hence, by our lemma, the equations

$$\frac{x}{a^2}-\lambda(x-\alpha)=0, \quad \frac{y}{b^2}-\lambda(y-\beta)=0, \quad \frac{z}{c^2}-\lambda(z-\gamma)=0$$

represent three planes through the line of intersection. The three planes pass through one line if

(i) $\lambda=\frac{1}{a^2}$, $\alpha=0$; or (ii) $\lambda=\frac{1}{b^2}$, $\beta=0$; or (iii) $\lambda=\frac{1}{c^2}$, $\gamma=0$.

The line is,

(i) $y=\frac{-b^2\beta}{a^2-b^2}$, $z=\frac{c^2\gamma}{c^2-a^2}$; or (ii) $z=\frac{-c^2\gamma}{b^2-c^2}$, $x=\frac{a^2\alpha}{a^2-b^2}$;

or (iii) $x=\frac{-a^2\alpha}{c^2-a^2}$, $y=\frac{b^2\beta}{b^2-c^2}$.

But the line is the line of intersection of the planes given by equation (1), and therefore the coordinates of any point on the line satisfy equation (1). Therefore, substituting from the equations to the line in equation (1), we obtain

(i) $\frac{-\beta^2}{a^2-b^2}-\frac{\gamma^2}{a^2-c^2}=1$, and in this case, $\alpha=0$;

(ii) $\frac{-\gamma^2}{b^2-c^2}+\frac{\alpha^2}{a^2-b^2}=1$, „ „ $\beta=0$;

(iii) $\frac{\alpha^2}{a^2-c^2}+\frac{\beta^2}{b^2-c^2}=1$, „ „ $\gamma=0$.

Hence, (i) there is an infinite number of imaginary foci in the yz-plane lying on the imaginary ellipse

$$x=0, \quad -\frac{y^2}{a^2-b^2}-\frac{z^2}{a^2-c^2}=1,$$

and the corresponding directrices are imaginary.

(ii) There is an infinite number of real foci in the zx-plane lying on the hyperbola

$$y=0, \quad \frac{x^2}{a^2-b^2}-\frac{z^2}{b^2-c^2}=1, \quad \text{(the **focal hyperbola**)},$$

and the corresponding directrices are real.

(iii) There is an infinite number of real foci in the xy-plane lying on the ellipse

$$z=0, \quad \frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}=1, \quad \text{(the **focal ellipse**)},$$

and the corresponding directrices are real

The directrix corresponding to a point $(\alpha, 0, \gamma)$ on the focal hyperbola has equations

$$x=\frac{a^2\alpha}{a^2-b^2}, \quad z=\frac{-c^2\gamma}{b^2-c^2},$$

and therefore, since $\dfrac{\alpha^2}{a^2-b^2}-\dfrac{\gamma^2}{b^2-c^2}=1,$

the directrices corresponding to points on the focal hyperbola lie on the hyperbolic cylinder

$$\frac{x^2(a^2-b^2)}{a^4}-\frac{z^2(b^2-c^2)}{c^4}=1.$$

Similarly, the directrices corresponding to foci which lie on the focal ellipse lie on the elliptic cylinder

$$\frac{x^2(a^2-c^2)}{a^4}+\frac{y^2(b^2-c^2)}{b^4}=1.$$

If $(\alpha, 0, \gamma)$ is a point on the focal hyperbola,

$$\mathsf{S}-\lambda\phi \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1-\frac{1}{b^2}\{(x-\alpha)^2+y^2+(z-\gamma)^2\},$$

$$=-\frac{a^2-b^2}{a^2b^2}\left(x-\frac{a^2\alpha}{a^2-b^2}\right)^2+\frac{b^2-c^2}{b^2c^2}\left(z+\frac{c^2\gamma}{b^2-c^2}\right)^2$$

$$+\frac{\alpha^2}{a^2-b^2}-\frac{\gamma^2}{b^2-c^2}-1,$$

$$=-\frac{a^2-b^2}{a^2b^2}(x-\xi)^2+\frac{b^2-c^2}{b^2c^2}(z-\zeta)^2,$$

where the equations to the directrix corresponding to $(\alpha, 0, \gamma)$ are $x=\xi$, $z=\zeta$. But the equations to the planes through the line $x=\xi$, $z=\zeta$, parallel to the real circular sections, are

$$\sqrt{\frac{a^2-b^2}{a^2}}(x-\xi)\pm\sqrt{\frac{b^2-c^2}{c^2}}(z-\zeta)=0.$$

Therefore any point on the focal hyperbola is a focus of the first species, and the ellipsoid is the locus of a point the square on whose distance from a focus of the first species is proportional to the rectangle under its distances from the two planes through the corresponding directrix parallel to the real circular sections.

If $(\alpha, \beta, 0)$ is a point on the focal ellipse,

$$\mathsf{S}-\lambda\phi \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1-\frac{1}{c^2}\{(x-\alpha)^2+(y-\beta)^2+z^2\},$$

$$=\frac{c^2-a^2}{a^2c^2}\left(x-\frac{a^2\alpha}{c^2-a^2}\right)^2+\frac{c^2-b^2}{b^2c^2}\left(y-\frac{b^2\beta}{b^2-c^2}\right)^2$$

$$+\frac{\alpha^2}{a^2-c^2}+\frac{\beta^2}{b^2-c^2}-1,$$

$$=\frac{c^2-a^2}{a^2c^2}(x-\xi)^2+\frac{c^2-b^2}{b^2c^2}(y-\eta)^2,$$

where the equations to the directrix corresponding to $(\alpha, \beta, 0)$ are $x=\xi$, $y=\eta$. Now the equation to a plane through P, (x', y', z') parallel to a real circular section is

$$\sqrt{\frac{a^2-b^2}{a^2}}(x-x')\pm\sqrt{\frac{b^2-c^2}{c^2}}(z-z')=0,$$

and hence this plane meets the directrix $x=\xi$, $y=\eta$ in the point P$'$, whose coordinates are

$$\xi,\quad \eta,\quad z'\pm\frac{c}{a}\sqrt{\frac{a^2-b^2}{b^2-c^2}}(\xi-x').$$

The distance PP$'$ is the distance of P from the directrix, measured parallel to the plane. It is given by

$$\left\{(x'-\xi)^2\frac{b^2(a^2-c^2)}{a^2(b^2-c^2)}+(y'-\eta)^2\right\}^{\frac{1}{2}},$$

or
$$\frac{bc}{\sqrt{b^2-c^2}}\left\{(x'-\xi)^2\frac{a^2-c^2}{a^2c^2}+(y'-\eta)^2\frac{b^2-c^2}{b^2c^2}\right\}^{\frac{1}{2}}.$$

Hence any point on the focal ellipse is a focus of the second species, and the ellipsoid is the locus of a point whose distance from a focus of the second species is proportional to its distance, measured parallel to a real circular section, from the corresponding directrix.

By the same methods, we find that the points on the parabolas

$$\text{(i) } x=0,\quad \frac{y^2}{a-b}=-2z+a;\qquad \text{(ii) } y=0,\quad \frac{x^2}{b-a}=-2z+b$$

are foci of the paraboloid $\dfrac{x^2}{a}+\dfrac{y^2}{b}=2z$. These parabolas are called the **focal parabolas**. The corresponding directrices generate the cylinders

$$\text{(i) } \frac{b-a}{b^2}y^2=2z+a,\quad \text{(ii) } \frac{a-b}{a^2}x^2=2z+b.$$

If $(0, \beta, \gamma)$ is any point on the focal parabola in the yz-plane,

$$S-\lambda\phi\equiv\frac{x^2}{a}+\frac{y^2}{b}-2z-\frac{1}{a}\{x^2+(y-\beta)^2+(z-\gamma)^2\}$$

$$=\frac{a-b}{ab}\left(y+\frac{b\beta}{a-b}\right)^2-\frac{1}{a}(z-\gamma+a)^2.$$

If $(\alpha, 0, \gamma)$ is any point on the focal parabola in the zx-plane,

$$S-\lambda\phi \equiv \frac{x^2}{a}+\frac{y^2}{b}-2z-\frac{1}{b}\{(x-\alpha)^2+y^2+(z-\gamma)^2\}$$
$$=\frac{b-a}{ab}\left(x+\frac{a\alpha}{b-a}\right)^2-\frac{1}{b}(z-\gamma+b)^2.$$

Whence the species of the foci can be determined if the signs and relative magnitudes of a and b are given.

Cor. All confocal conicoids have the same focal conics.

Ex. 1. Prove that the product of the eccentricities of the focal conics is unity.

Ex. 2. Find the equations to the focal conics of the hyperboloid $x^2+yz-2=0$. *Ans.* $x=0$, $y^2+4yz+z^2=6$; $y-z=0$, $2x^2+3y^2=12$.

Ex. 3. If P is a point on a focal conic, the corresponding directrix intersects the normal at P to the conic.

Ex. 4. If P is a point on a focal conic the section of the conicoid by the plane through P at right angles to the tangent at P to the conic has a focus at P.

Ex. 5. If P is any point on the directrix of a conicoid which corresponds to a focus S, the polar plane of P passes through S and is at right angles to SP.

Ex. 6. The polar plane of any point A cuts the directrix corresponding to a focus S at the point P. Prove that AS is at right angles to SP.

Ex. 7. If the normal and tangent plane at any point P of a conicoid meet a principal plane in the point N and the line QR, QR is the polar of N with respect to the focal conic that lies in the principal plane.

Ex. 8. Prove that the real foci of a cone lie upon two straight lines through the vertex (the **focal lines**).

Ex. 9. Prove that the focal lines of a cone are normal to the cyclic planes of the reciprocal cone.

Ex. 10. The enveloping cones with vertex P of a system of confocal conicoids have the same focal lines, and the focal lines are the generators of the confocal hyperboloid of one sheet that passes through P.

*Examples VII.

1. If the enveloping cone of an ellipsoid has three mutually perpendicular generators the plane of contact envelopes a confocal.

2. The locus of the polars of a given line with respect to a system of confocals is a hyperbolic paraboloid.

3. Through a straight line in one of the principal planes, tangent planes are drawn to a system of confocals. Prove that the points of contact lie in a plane and that the normals at these points pass through a fixed point in the principal plane.

4. Shew that the locus of the centres of the sections of a system of confocals by a given plane is a straight line.

5. If PQ is perpendicular to its polar with respect to an ellipsoid, it is perpendicular to its polars with respect to all confocal ellipsoids.

6. Any tangent plane to a cone makes equal angles with the planes through the generator of contact and the focal lines.

7. Through any tangent to a conicoid two planes are drawn to touch a confocal. Prove that they are equally inclined to the tangent plane to the conicoid that contains the tangent.

8. The locus of the intersection of three mutually perpendicular planes each of which touches a confocal is a sphere.

9. The sum of the angles that any generator of a cone makes with the focal lines is constant.

10. The four planes through two generators OP and OQ of a cone and the focal lines touch a right circular cone whose axis is the line of intersection of the tangent planes which touch the cone along OP and OQ.

11. The planes which bisect the angles between two tangent planes to a cone also bisect the angles between the planes containing their line of intersection and the focal lines.

12. A conicoid of revolution is formed by the revolution of an ellipse whose foci are S and S′. Prove that the focal lines of the enveloping cone whose vertex is P are PS and PS′.

13. The feet of the normals to a system of confocals which are parallel to a fixed line lie on a rectangular hyperbola one of whose asymptotes is parallel to the line.

14. A tangent plane to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ intersects the two confocals whose parameters are λ and μ. Prove that the enveloping cones to the confocals along the curves of section have a common section which lies on the conicoid

$$\frac{a^2x^2}{(a^2-\lambda)(a^2-\mu)}+\frac{b^2y^2}{(b^2-\lambda)(b^2-\mu)}+\frac{c^2z^2}{(c^2-\lambda)(c^2-\mu)}=1.$$

15. The three principal planes intercept on any normal to a confocal of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, two segments whose ratio is constant. Also the normals to the confocals which lie in a given plane $lx+my+nz=0$ are parallel to the line

$$\frac{lx}{b^2-c^2}=\frac{my}{c^2-a^2}=\frac{nz}{a^2-b^2}.$$

16. The cone that contains the normals from P to a conicoid contains the normals from P to all the confocals, and its equation referred to the normals to the confocals through P as coordinate axes is

$$\frac{p_1(\lambda_2-\lambda_3)}{x}+\frac{p_2(\lambda_3-\lambda_1)}{y}+\frac{p_3(\lambda_1-\lambda_2)}{z}=0.$$

17. Normals are drawn from a point in one of the principal planes to a system of confocals. Prove that they lie in the principal plane or in a plane at right angles to it, that the tangent planes at the feet of those in the principal plane touch a parabolic cylinder, and that the tangent planes at the feet of the others pass through a straight line lying in the principal plane.

18. If tangent planes are drawn through a fixed line to a system of confocals the normals at the points of contact generate a hyperbolic paraboloid. Shew that the paraboloid degenerates into a plane when the given line is a normal to one of the surfaces of the system.

19. From any two fixed points on the same normal to an ellipsoid perpendiculars are drawn to their respective polar planes with regard to any confocal ellipsoid. Prove that the perpendiculars intersect and that the locus of their intersection as the confocal varies is a cubic curve whose projection on any principal plane is a rectangular hyperbola.

20. Find the parabola which is the envelope of the normals to the confocals $\dfrac{x^2}{a^2+\lambda}+\dfrac{y^2}{b^2+\lambda}+\dfrac{z^2}{c^2+\lambda}=1$ which lie in the plane $lx+my+nz=p$, and prove that its directrix lies in the plane

$$(b^2-c^2)x/l+(c^2-a^2)y/m+(a^2-b^2)z/n=0.$$

21. If λ, μ, ν are the direction-cosines of the normal to a system of parallel tangent planes to a system of confocal conicoids, express the coordinates of any point of the locus of their points of contact in the form

$$x=\lambda(t+a^2/t),\quad y=\mu(t+b^2/t),\quad z=\nu(t+c^2/t),$$

where a, b, c are the principal axes of a particular confocal of the system. Deduce that the locus is a rectangular hyperbola.

22. If λ, μ, ν are the parameters of the confocals of an ellipsoid, axes a, b, c, through a point P, the perpendicular from P to its polar plane is of length

$$\lambda\mu\nu\{b^2c^2\mu\nu(a^2-\lambda)+c^2a^2\nu\lambda(b^2-\mu)+a^2b^2\lambda\mu(c^2-\nu)\}^{-\frac{1}{2}}.$$

23. Through a given line tangent planes are drawn to two confocals and touch them in A, A′; B, B′ respectively. Shew that the lines AB, AB′ are equally inclined to the normal at A and are coplanar with it.

24. If P and Q are points on two confocals such that the tangent planes at P and Q are at right angles, the plane through the centre and the line of intersection of the tangent planes bisects PQ. Hence shew that if a conicoid touches each of three given confocals at two points it has a fixed director sphere.

CHAPTER XI.

THE GENERAL EQUATION OF THE SECOND DEGREE.

131. In Chapter VII. we have found the equations to certain loci, (tangent planes, polar planes, etc.) connected with the conicoid, when the conicoid is represented by an equation referred to conjugate diametral planes as coordinate planes. We shall in this chapter first find the equations to these loci when the conicoid is represented by the general equation of the second degree, and then discuss the determination of the centre and principal planes, and the transformation of the equation when the principal planes are taken as coordinate planes.

132. Constants in equation of second degree. The general equation of the second degree may be written

$$\mathsf{F}(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$
$$+ 2ux + 2vy + 2wz + d = 0,$$

or
$$f(x, y, z) + 2ux + 2vy + 2wz + d = 0.$$

It contains nine disposable constants, and therefore a conicoid can be found to satisfy nine conditions which each involve one relation between the constants; *e.g.* a conicoid can be found to pass through nine given points no four of which are coplanar, or to pass through six given points and touch the plane XOY at the origin, or to pass through three given non-intersecting lines.

Ex. 1. A conicoid is to pass through a given conic. How many disposable constants will its equation contain? Is the number the same when the conicoid is to pass through a given circle?

Ex. 2. The equation to a conicoid through the conic $z=0$, $\phi=0$, is
$$\phi+z(ax+by+cz+d)=0,$$
where a, b, c, d are disposable constants.

Ex. 3. The equation to a conicoid that touches the plane $z=0$ at an umbilic at the origin and touches the plane $lx+my+nz=p$ is
$$z(lx+my+nz-p)+(\lambda x+\mu z)^2+(\lambda y+\nu z)^2=0,$$
where λ, μ, ν are disposable constants.

Ex. 4. Find the equation to the conicoid which passes through the circle $x^2+y^2=2ax$, $z=0$, and the points $(b, 0, c)$, $(0, b, c)$, and has the z-axis as a generator.

Ans. $c(x^2+y^2-2ax)-byz+(2a-b)zx=0$.

133. Points of intersection of line and conicoid. The equations to the straight line through A, (α, β, γ), whose direction-ratios are l, m, n, are
$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}, \quad \text{.....................(1)}$$
and the point on this line whose distance from A is r has coordinates $\alpha+lr$, $\beta+mr$, $\gamma+nr$. It lies on the conicoid
$$F(x, y, z)=0,$$
if
$$F(\alpha+lr,\ \beta+mr,\ \gamma+nr)=0:$$
that is, if
$$F(\alpha, \beta, \gamma)+r\left(l\frac{\partial F}{\partial \alpha}+m\frac{\partial F}{\partial \beta}+n\frac{\partial F}{\partial \gamma}\right)+r^2f(l, m, n)=0. \quad \text{......(2)}$$

Hence the straight line meets the conicoid in two points P and Q, and the measures of AP and AQ are the roots of the equation (2).

If (i) $F(\alpha, \beta, \gamma)=0$, (ii) $l\frac{\partial F}{\partial \alpha}+m\frac{\partial F}{\partial \beta}+n\frac{\partial F}{\partial \gamma}=0$,

(iii) $f(l, m, n)=0$,

equation (2) is satisfied by all values of r, or every point on the line lies on the conicoid. The conditions (ii) and (iii) give two sets of values for $l:m:n$, and therefore through any point on a conicoid two straight lines can be drawn to lie wholly on the conicoid. They are parallel to the lines in which the plane
$$x\frac{\partial F}{\partial \alpha}+y\frac{\partial F}{\partial \beta}+z\frac{\partial F}{\partial \gamma}=0$$

cuts the cone $f(x, y, z)=0$, (cf. § 60). They may be real, imaginary, or coincident, as in the cases of the hyperboloid of one sheet, the ellipsoid, and the cone, respectively.

134. The tangent plane. If $F(\alpha, \beta, \gamma)=0$, A is on the conicoid, and one root of equation (2) is zero. A coincides with P or with Q. If also

$$l\frac{\partial F}{\partial \alpha}+m\frac{\partial F}{\partial \beta}+n\frac{\partial F}{\partial \gamma}=0, \quad \text{.....................(3)}$$

both roots of equation (2) are zero, and P and Q coincide at A, which lies on the conicoid. The line is therefore a tangent line to the surface at A. If we eliminate l, m, n between the equations to the line and equation (3), we obtain the equation to the locus of the tangent lines drawn through A in all possible directions. The equation is

$$(x-\alpha)\frac{\partial F}{\partial \alpha}+(y-\beta)\frac{\partial F}{\partial \beta}+(z-\gamma)\frac{\partial F}{\partial \gamma}=0,$$

and hence the locus is a plane, the **tangent plane** at A. The above equation may be written

$$x\frac{\partial F}{\partial \alpha}+y\frac{\partial F}{\partial \beta}+z\frac{\partial F}{\partial \gamma}=\alpha\frac{\partial F}{\partial \alpha}+\beta\frac{\partial F}{\partial \beta}+\gamma\frac{\partial F}{\partial \gamma}. \quad \text{.........(4)}$$

If, now, $F(x, y, z)$ be made homogeneous by the introduction of an auxiliary variable t, which is equated to unity after differentiation, equation (4) is equivalent to

$$\begin{aligned} x\frac{\partial F}{\partial \alpha}+y\frac{\partial F}{\partial \beta}+z\frac{\partial F}{\partial \gamma}+t\frac{\partial F}{\partial t}&=\alpha\frac{\partial F}{\partial \alpha}+\beta\frac{\partial F}{\partial \beta}+\gamma\frac{\partial F}{\partial \gamma}+t\frac{\partial F}{\partial t}, \\ &=2F(\alpha, \beta, \gamma, t), \text{ (Euler's Theorem)}, \\ &=0. \end{aligned}$$

Ex. 1. Find the equations to the tangent planes at (x', y', z') on

(i) $xy=cz$, (ii) $x^2+2yz=a^2$.

Ans. (i) $x/x'+y/y'-z/z'=1$, (ii) $xx'+yz'+zy'=a^2$.

Ex. 2. The bisectors of the angles between the lines in which any tangent plane to $z^2=4xy$ meets the planes $x=0$, $y=0$, lie in the planes $x+y+z=0$, $x+y-z=0$.

Ex. 3. Find the equation to the tangent plane at (1, 2, 3) on the hyperboloid

$$x^2+8y^2+z^2-9yz+14zx-16xy-6x-y+4z-2=0,$$

and the equations to the two generators through the point.

Ans. (i) $x-2y+z=0$; (ii) the equation (i) and $4x-3y+2=0$, $3x-2y+1=0$.

Ex. 4. Find the condition that the plane $lx+my+nz+p=0$ should touch the conicoid $F(x, y, z)=0$

If the point of contact is (α, β, γ), then

$$x\frac{\partial F}{\partial \alpha}+y\frac{\partial F}{\partial \beta}+z\frac{\partial F}{\partial \gamma}+t\frac{\partial F}{\partial t}=0, \quad \text{and} \quad lx+my+nz+p=0$$

represent the same plane. Therefore

$$\frac{\dfrac{\partial F}{\partial \alpha}}{l}=\frac{\dfrac{\partial F}{\partial \beta}}{m}=\frac{\dfrac{\partial F}{\partial \gamma}}{n}=\frac{\dfrac{\partial F}{\partial t}}{p}=-2\lambda, \text{ say.}$$

Hence

$$\begin{aligned} a\alpha+h\beta+g\gamma+u+l\lambda&=0,\\ h\alpha+b\beta+f\gamma+v+m\lambda&=0,\\ g\alpha+f\beta+c\gamma+w+n\lambda&=0,\\ u\alpha+v\beta+w\gamma+d+p\lambda&=0.\end{aligned}$$

And

$$l\alpha+m\beta+n\gamma+p=0,$$

since the point of contact must lie in the given plane. Therefore eliminating $\alpha, \beta, \gamma, \lambda$, we obtain the required condition, viz.:

$$\begin{vmatrix} a, & h, & g, & u, & l \\ h, & b, & f, & v, & m \\ g, & f, & c, & w, & n \\ u, & v, & w, & d, & p \\ l, & m, & n, & p, & 0 \end{vmatrix}=0.$$

Ex. 5. Prove that $lx+my+nz=p$ touches $xy=cz$ if $clm+np=0$.

Ex. 6. Prove that $lx+my+nz=p$ touches $f(x, y, z)=1$ if

$$Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=p^2D,$$

where $$D\equiv\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}, \text{ and } A=\frac{\partial D}{\partial a}, \ B=\frac{\partial D}{\partial b}, \text{ etc.}$$

Ex. 7. If the axes are rectangular, prove that the locus of the feet of the perpendiculars from the origin to tangent planes to $f(x, y, z)=1$ is

$$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=D(x^2+y^2+z^2)^2.$$

Ex. 8. Prove that the locus of the point of intersection of three mutually perpendicular tangent planes to $f(x, y, z)=1$ is the sphere

$$D(x^2+y^2+z^2)=A+B+C. \quad \text{(Cf. § 68, Ex. 1.)}$$

Ex. 9. Prove that the plane $2y-2z=1$ is a tangent plane to the surface
$$x^2+7y^2+2z^2-9yz+5zx-6xy+5x-14y+10z+6=0.$$

Prove also that the lines of intersection of the given plane and the planes $2x+3=0$, $2x-2z+1=0$ lie on the surface.

Ex. 10. If two conicoids have a common generator, they touch at two points of the generator.

If the generator is taken as x-axis, the equations to the conicoids are
$$by^2+cz^2+2fyz+2gzx+2hxy+2vy+2wz=0,$$
$$b'y^2+c'z^2+2f'yz+2g'zx+2h'xy+2v'y+2w'z=0.$$

The tangent planes at $(\alpha, 0, 0)$ are
$$y(h\alpha+v)+z(g\alpha+w)=0, \quad y(h'\alpha+v')+z(g'\alpha+w')=0.$$

They are coincident if
$$\frac{h\alpha+v}{h'\alpha+v'}=\frac{g\alpha+w}{g'\alpha+w'}.$$

This equation gives two values of α.

Ex. 11. If two conicoids touch at three points of a common generator, they touch at all points of the generator, and the generator has the same central point and parameter of distribution for both surfaces.

Ex. 12. Tangent planes parallel to the given plane
$$\alpha x+\beta y+\gamma z=0$$
are drawn to conicoids that pass through the lines $x=0$, $y=0$; $z=0$, $x=c$. Shew that the points of contact lie on the paraboloid
$$x(\alpha x+\beta y+\gamma z)=c(\alpha x+\beta y).$$

Ex. 13. If a conicoid passes through the origin, and the tangent plane at the origin is taken as $z=0$, the equation to the surface is
$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2wz=0.$$

Ex. 14. If a set of rectangular axes through a fixed point O of a conicoid meet the conicoid in P, Q, R, the plane PQR meets the normal at O in a fixed point.

Ex. 15. If $\quad u_r \equiv a_r x+b_r y+c_r z+d_r, \quad r=1, 2, 3,$

prove that the tangent planes at (x', y', z') to the conicoids
$$\text{(i) } \lambda_1 u_1{}^2+\lambda_2 u_2{}^2+\lambda_3 u_3{}^2=1,$$
$$\text{(ii) } \lambda_1 u_1{}^2+\lambda_2 u_2{}^2=2\lambda_3 u_3$$
are given by
$$\text{(i) } \lambda_1 u_1 u_1'+\lambda_2 u_2 u_2'+\lambda_3 u_3 u_3'=1,$$
$$\text{(ii) } \lambda_1 u_1 u_1'+\lambda_2 u_2 u_2'=\lambda_3(u_3+u_3'),$$
where
$$u_r' \equiv a_r x'+b_r y'+c_r z'+d_r.$$

Ex. 16. Prove that
$$u_1 u_2+\lambda_1 v_1{}^2+2\lambda_2 v_1 v_2+\lambda_3 v_2{}^2=0$$
represents a conicoid touching the planes $u_1=0$, $u_2=0$ at their points of intersection with the line $v_1=0=v_2$.

135. The polar plane. If any secant through A meets the conicoid in P and Q and if R is the harmonic conjugate of A with respect to P and Q, the locus of R is the polar of A.

If A is (α, β, γ) and the equations to the secant are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n},$$

then r_1, r_2, the measures of AP and AQ, are the roots of the equation

$$r^2 f(l, m, n)+r\left(l\frac{\partial F}{\partial \alpha}+m\frac{\partial F}{\partial \beta}+n\frac{\partial F}{\partial \gamma}\right)+F(\alpha, \beta, \gamma)=0.$$

Hence if R is (ξ, η, ζ) and the measure of AR is ρ,

$$\rho=\frac{2r_1 r_2}{r_1+r_2}=-\frac{2F(\alpha, \beta, \gamma)}{l\frac{\partial F}{\partial \alpha}+m\frac{\partial F}{\partial \beta}+n\frac{\partial F}{\partial \gamma}},$$

and $\qquad \xi-\alpha=l\rho, \quad \eta-\beta=m\rho, \quad \zeta-\gamma=n\rho.$

Therefore

$$(\xi-\alpha)\frac{\partial F}{\partial \alpha}+(\eta-\beta)\frac{\partial F}{\partial \beta}+(\zeta-\gamma)\frac{\partial F}{\partial \gamma}=-2F(\alpha, \beta, \gamma),$$

and the equation to the locus of (ξ, η, ζ), the **polar plane**, becomes

$$\begin{aligned}
x\frac{\partial F}{\partial \alpha}+y\frac{\partial F}{\partial \beta}+z\frac{\partial F}{\partial \gamma}+t\frac{\partial F}{\partial t}
&=\alpha\frac{\partial F}{\partial \alpha}+\beta\frac{\partial F}{\partial \beta}+\gamma\frac{\partial F}{\partial \gamma}+t\frac{\partial F}{\partial t}-2F(\alpha, \beta, \gamma, t),\\
&=2F(\alpha, \beta, \gamma, t)-2F(\alpha, \beta, \gamma, t),\\
&=0.
\end{aligned}$$

Ex. 1. Find the equations to the polar of $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ with respect to the conicoid $F(x, y, z)=0$. (Cf. § 70.)

Ans. $x\frac{\partial F}{\partial \alpha}+y\frac{\partial F}{\partial \beta}+z\frac{\partial F}{\partial \gamma}+t\frac{\partial F}{\partial t}=0, \quad l\frac{\partial F}{\partial x}+m\frac{\partial F}{\partial y}+n\frac{\partial F}{\partial z}=0.$

Ex. 2. Prove that the lines

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}, \quad \frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}$$

are polar with respect to the conicoid $F(x, y, z)=0$ if

$$\alpha'\frac{\partial F}{\partial \alpha}+\beta'\frac{\partial F}{\partial \beta}+\gamma'\frac{\partial F}{\partial \gamma}+\frac{\partial F}{\partial t}=0, \qquad l'\frac{\partial F}{\partial \alpha}+m'\frac{\partial F}{\partial \beta}+n'\frac{\partial F}{\partial \gamma}=0,$$

$$l\frac{\partial F}{\partial \alpha'}+m\frac{\partial F}{\partial \beta'}+n\frac{\partial F}{\partial \gamma'}=0, \text{ and } l'\frac{\partial f}{\partial l}+m'\frac{\partial f}{\partial m}+n'\frac{\partial f}{\partial n}=0.$$

Ex. 3. Any set of rectangular axes through a fixed point O meets a given conicoid in six points. Prove that the sum of the squares of the ratios of the distances of the points from the polar plane of O to their distances from O is constant.

(Take O as origin, and use § 54, Ex. 9.)

Ex. 4. Prove that

$$\lambda_1 u_1^2+\lambda_2 u_2^2+\lambda_3 u_3^2+\lambda_4 u_4^2=0$$

represents a conicoid with respect to which the tetrahedron whose faces are $u_1=0$, $u_2=0$, $u_3=0$, $u_4=0$ is self-conjugate.

Ex. 5. Find the equation to the conicoid with respect to which the tetrahedron formed by the coordinate planes and the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

is self-conjugate, and which passes through the points $(-a, 0, 0)$, $(0, -b, 0)$, $(0, 0, -c)$.

Ans. $4\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)-\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1\right)^2=0.$

Ex. 6. All conicoids which touch a given cone at its points of section by a given plane have a common self-conjugate tetrahedron.

136. The enveloping cone. The equations

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

represent a tangent if equation (2) of § 133 has equal roots. The condition for equal roots is

$$4F(\alpha, \beta, \gamma)f(l, m, n)=\left(l\frac{\partial F}{\partial \alpha}+m\frac{\partial F}{\partial \beta}+n\frac{\partial F}{\partial \gamma}\right)^2.$$

Therefore the equation to the locus of the tangents drawn from a given point (α, β, γ) is

$$4F(\alpha, \beta, \gamma)f(x-\alpha, y-\beta, z-\gamma)$$
$$=\left\{(x-\alpha)\frac{\partial F}{\partial \alpha}+(y-\beta)\frac{\partial F}{\partial \beta}+(z-\gamma)\frac{\partial F}{\partial \gamma}\right\}^2.$$

But $\quad F(\alpha+\overline{x-\alpha}, \beta+\overline{y-\beta}, \gamma+\overline{z-\gamma})$

$$=f(x-\alpha, y-\beta, z-\gamma)$$
$$+(x-\alpha)\frac{\partial F}{\partial \alpha}+(y-\beta)\frac{\partial F}{\partial \beta}+(z-\gamma)\frac{\partial F}{\partial \gamma}+F(\alpha, \beta, \gamma).$$

Therefore the equation to the locus becomes

$$4F(\alpha, \beta, \gamma)F(x, y, z)$$
$$=\left\{(x-\alpha)\frac{\partial F}{\partial \alpha}+(y-\beta)\frac{\partial F}{\partial \beta}+(z-\gamma)\frac{\partial F}{\partial \gamma}+2F(\alpha, \beta, \gamma)\right\}^2,$$
$$=\left(x\frac{\partial F}{\partial \alpha}+y\frac{\partial F}{\partial \beta}+z\frac{\partial F}{\partial \gamma}+t\frac{\partial F}{\partial t}\right)^2.$$

Ex. 1. If a cone envelope a sphere, the section of the cone by any tangent plane to the sphere is a conic which has a focus at the point of contact.

Ex. 2. The tangent plane to a conicoid at an umbilic meets any enveloping cone in a conic of which the umbilic is a focus.

Ex. 3. Find the locus of a luminous point which moves so that the sphere

$$x^2+y^2+z^2-2az=0$$

casts a parabolic shadow on the plane $z=0$.

Ans. $z=2a$.

137. The enveloping cylinder. From the condition for equal roots used in the last paragraph we see that (α, β, γ), any point on a tangent drawn parallel to the fixed line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n},$$

lies on the cylinder given by

$$4F(x, y, z)f(l, m, n)=\left(l\frac{\partial F}{\partial x}+m\frac{\partial F}{\partial y}+n\frac{\partial F}{\partial z}\right)^2.$$

Ex. A cylinder whose generators make an angle α with the z-axis envelopes the sphere $x^2+y^2+z^2=2az$. Prove that the eccentricity of its section by the plane $z=0$ is $\sin\alpha$.

138. The locus of the chords which are bisected at a given point. If (α, β, γ), the given point, is the mid-point of the chord whose equations are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n},$$

the equation

$$F(\alpha, \beta, \gamma)+r\left(l\frac{\partial F}{\partial \alpha}+m\frac{\partial F}{\partial \beta}+n\frac{\partial F}{\partial \gamma}\right)+r^2f(l, m, n)=0$$

takes the form $r^2=k^2$, and therefore

$$l\frac{\partial F}{\partial \alpha}+m\frac{\partial F}{\partial \beta}+n\frac{\partial F}{\partial \gamma}=0. \quad \text{..................(1)}$$

Hence all chords which are bisected at (α, β, γ) lie in the plane given by

$$(x-\alpha)\frac{\partial F}{\partial \alpha}+(y-\beta)\frac{\partial F}{\partial \beta}+(z-\gamma)\frac{\partial F}{\partial \gamma}=0.$$

The section of the conicoid by this plane is a conic of which (α, β, γ) is the centre.

Ex. Find the locus of centres of sections of

$$ayz+bzx+cxy+abc=0$$

which touch $\qquad x^2/\alpha^2+y^2/\beta^2+z^2/\gamma^2=1.$

Ans. $\alpha^2(bz+cy)^2+\beta^2(cx+az)^2+\gamma^2(ay+bx)^2=4(ayz+bzx+cxy)^2.$

139. The diametral plane. Equation (1) of § 138 shews that the mid-points of all chords drawn parallel to a fixed line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

lie on the diametral plane whose equation is

$$l\frac{\partial F}{\partial x}+m\frac{\partial F}{\partial y}+n\frac{\partial F}{\partial z}=0.$$

Ex. 1. Find the central circular sections and umbilics of the following surfaces:

(i) $x^2+yz+2=0,$

(ii) $4yz+5zx-5xy+8=0,$

(iii) $y^2-yz-2zx-xy-4=0.$

Ans. (i) $x+y-z=0,\ x-y+z=0;$

$$\frac{x}{1}=\frac{y}{-2}=\frac{z}{2}=\pm\sqrt{\frac{2}{3}},\quad \frac{x}{1}=\frac{y}{2}=\frac{z}{-2}=\pm\sqrt{\frac{2}{3}}.$$

(ii) $2x+y-z=0,\ x+2y-2z=0;$

$$\frac{x}{1}=\frac{y}{5}=\frac{z}{-5}=\pm\frac{2\sqrt{3}}{15},\quad \frac{x}{16}=\frac{y}{5}=\frac{z}{-5}=\pm\frac{\sqrt{2}}{15}.$$

(iii) $x+z=0,\ x+y+z=0;$ the umbilics are imaginary.

Ex. 2. Prove that the umbilics of conicoids that pass through the circles

$$z=0,\ x^2+y^2=a^2;\quad x=0,\ y^2+z^2=a^2$$

lie on two equal hyperbolas in the zx-plane.

140. The principal planes. A diametral plane which is at right angles to the chords which it bisects is a principal

plane. If the axes are rectangular, the diametral plane whose equation is

$$l\frac{\partial F}{\partial x}+m\frac{\partial F}{\partial y}+n\frac{\partial F}{\partial z}=0,$$

or $$x(al+hm+gn)+y(hl+bm+fn)+z(gl+fm+cn)$$
$$+ul+vm+wn=0,$$

is at right angles to the line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n},$$

if $$\frac{al+hm+gn}{l}=\frac{hl+bm+fn}{m}=\frac{gl+fm+cn}{n}.$$

If each of these ratios is equal to λ, then

$$\left.\begin{aligned}(a-\lambda)l+hm+gn=0,\\ hl+(b-\lambda)m+fn=0,\\ gl+fm+(c-\lambda)n=0.\end{aligned}\right\}\quad\text{.....................(1)}$$

Therefore λ is a root of the equation

$$\begin{vmatrix} a-\lambda, & h, & g \\ h, & b-\lambda, & f \\ g, & f, & c-\lambda \end{vmatrix}=0,$$

or $$\lambda^3-\lambda^2(a+b+c)+\lambda(bc+ca+ab-f^2-g^2-h^2)-D=0,$$

where $D\equiv\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c\end{vmatrix}$. Or if $A\equiv\dfrac{\partial D}{\partial a}$, etc., the equation may be written

$$\lambda^3-\lambda^2(a+b+c)+\lambda(A+B+C)-D=0.$$

This equation is called the **Discriminating Cubic**. It gives three values of λ, to each of which corresponds a set of values for $l:m:n$, and by substituting these sets in the equation

$$l\frac{\partial F}{\partial x}+m\frac{\partial F}{\partial y}+n\frac{\partial F}{\partial z}=0,$$

which, by means of the relations (1), reduces to

$$\lambda(lx+my+nz)+ul+vm+wn=0, \text{............(2)}$$

we obtain the equations to the three principal planes of the conicoid.

Ex. Find the principal planes of the conicoids:

(i) $14x^2+14y^2+8z^2-4yz-4zx-8xy+18x-18y+5=0$,

(ii) $3x^2+5y^2+3z^2-2yz+2zx-2xy+2x+12y+10z+20=0$.

Ans. (i) $\lambda=6, 12, 18$; $x+y+2z=0$, $x+y-z=0$, $x-y+1=0$;

(ii) $\lambda=2, 3, 6$; $x-z-2=0$, $x+y+z+4=0$, $x-2y+z-1=0$.

141. Condition for two zero-roots. If $\mathsf{D}=0$, then

$$\mathsf{BC}-\mathsf{F}^2=a\mathsf{D}=0, \quad \mathsf{CA}-\mathsf{G}^2=b\mathsf{D}=0, \quad \mathsf{AB}-\mathsf{H}^2=c\mathsf{D}=0,$$

and therefore A, B, C have the same sign. Therefore if $\mathsf{D}=0$ and $\mathsf{A}+\mathsf{B}+\mathsf{C}=0$, $\mathsf{A}=\mathsf{B}=\mathsf{C}=0$, and therefore we have also $\mathsf{F}=\mathsf{G}=\mathsf{H}=0$. Hence if the discriminating cubic has two zero-roots, all the six quantities A, B, C, F, G, H, are zero and $f(x, y, z)$ is a perfect square.

142. Case of one zero-root. If the discriminating cubic has one zero-root, the corresponding principal plane either is at an infinite distance or may be any plane at right angles to a fixed line. For if $\lambda=0$, the equations (1), § 140, give

$$\frac{l}{\mathsf{G}}=\frac{m}{\mathsf{F}}=\frac{n}{\mathsf{C}},$$

or

$$\frac{l}{\sqrt{\mathsf{A}}}=\frac{m}{\sqrt{\mathsf{B}}}=\frac{n}{\sqrt{\mathsf{C}}}, \quad (\S\ 141).$$

These determine a fixed direction, since A, B, C are not all zero. The corresponding principal plane has, by § 140, (2), the equation

$$\sqrt{\mathsf{A}}x+\sqrt{\mathsf{B}}y+\sqrt{\mathsf{C}}z+\frac{\sqrt{\mathsf{A}}u+\sqrt{\mathsf{B}}v+\sqrt{\mathsf{C}}w}{0}=0,$$

and is at an infinite distance if $\sqrt{\mathsf{A}}u+\sqrt{\mathsf{B}}v+\sqrt{\mathsf{C}}w\neq 0$, or may be any plane at right angles to the fixed line

$$\frac{x}{\sqrt{\mathsf{A}}}=\frac{y}{\sqrt{\mathsf{B}}}=\frac{z}{\sqrt{\mathsf{C}}}, \text{ if } \sqrt{\mathsf{A}}u+\sqrt{\mathsf{B}}v+\sqrt{\mathsf{C}}w=0.$$

In the first case the conicoid is a paraboloid whose axis is in the fixed direction, in the second, an elliptic or hyperbolic cylinder or pair of intersecting planes whose axis or line of intersection is in the fixed direction.

Ex. 1. Find the principal planes of the surfaces

(i) $2x^2+20y^2+18z^2-12yz+12xy+22x+6y-2z-2=0$,

(ii) $5x^2+26y^2+10z^2+4yz+14zx+6xy-8x-18y-10z+4=0$.

Ans. (i) $\lambda=14, 26, 0$; $x+2y+3z+1=0$, $x+4y-3z+1=0$, the plane at infinity:

(ii) $\lambda=14, 27, 0$; $2x-y+3z=1$, $x+5y+z=2$, any plane at right angles to $\dfrac{x}{-16}=\dfrac{y}{1}=\dfrac{z}{11}$.

Ex 2. Verify that the principal planes in Exs. §§ 140, 142 are mutually at right angles.

143. Case of two zero-roots. If the discriminating cubic has two zero-roots the equations (1), § 140, when $\lambda=0$, all reduce to

$$\sqrt{a}l+\sqrt{b}m+\sqrt{c}n=0,$$

and therefore the directions of the normals to two of the principal planes are indeterminate. These planes, however, must be at right angles to the plane $\sqrt{a}x+\sqrt{b}y+\sqrt{c}z=0$, and they may be at an infinite distance, (if $ul+vm+wn\neq 0$), or at any distance from the origin, (if $ul+vm+wn=0$). In the first case the surface is a parabolic cylinder and the axes of normal sections are parallel to the plane

$$\sqrt{a}x+\sqrt{b}y+\sqrt{c}z=0;$$

in the second the surface is a pair of planes parallel to

$$\sqrt{a}x+\sqrt{b}y+\sqrt{c}z=0.$$

Ex. 1. For the surfaces

(i) $x^2+y^2+z^2-2yz+2zx-2xy-2x-4y-2z+3=0$,

(ii) $x^2+y^2+z^2-2yz+2zx-2xy-2x+2y-2z-3=0$,

$\lambda=3, 0, 0$. The determinate principal plane is $x-y+z=0$. If

$$x-y+z=0, \quad x+2y+z=0, \quad x-z=0$$

are taken as coordinate planes the equations transform into

$$3\xi^2=2\sqrt{6}\eta-3, \quad 3\xi^2-2\sqrt{3}\xi-3=0.$$

Ex. 2. If one of the principal planes of the cone whose vertex is P and base the parabola $y^2=4ax$, $z=0$ is parallel to the fixed plane

$$lx+my+nz=0,$$

the locus of P is the straight line

$$\frac{z}{n}-\frac{y}{m}=\frac{2a}{l}, \quad \frac{2x}{m}+\frac{y}{l}+\frac{z}{m}\left(\frac{n}{l}-\frac{l}{n}\right)=0.$$

THE DISCRIMINATING CUBIC.

144. *All the roots of the discriminating cubic are real.*

The equation may be written,

$$\phi(\lambda) \equiv (\lambda - a)\{(\lambda - b)(\lambda - c) - f^2\}$$
$$- \{(\lambda - b)g^2 + (\lambda - c)h^2 + 2fgh\} = 0.$$

We may assume $a > b > c$. Consider

$$y = \psi(\lambda) \equiv (\lambda - b)(\lambda - c) - f^2.$$

Corresponding values of λ and y are

$$\begin{array}{cccc} -\infty, & c, & b, & +\infty, \\ +\infty, & -f^2, & -f^2, & +\infty. \end{array}$$

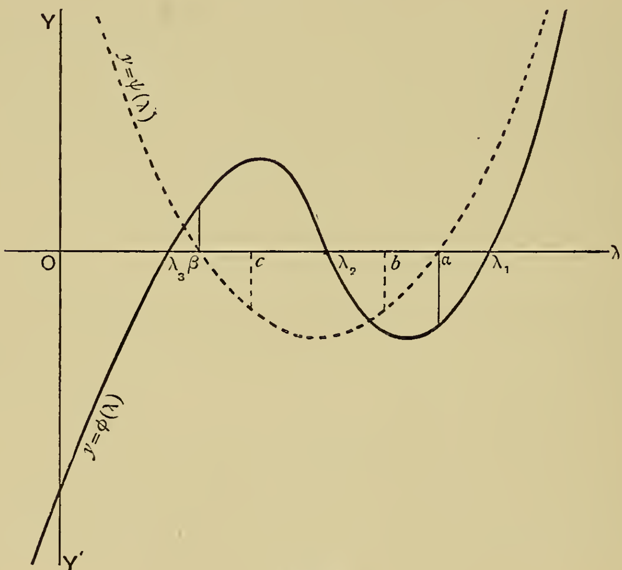

FIG. 48.

Hence from the graph, (fig. 48), it appears that the equation $\psi(\lambda) = 0$ has real roots α and β, such that

$$\beta < c < b < \alpha.$$

Consider now $y = \phi(\lambda)$. When $\lambda = \pm\infty$, $y = \pm\infty$;

when

$$\lambda=\alpha,\quad y=-\{(\alpha-b)g^2+(\alpha-c)h^2\pm 2\sqrt{(\alpha-b)(\alpha-c)}gh\}$$
$$=-(g\sqrt{\alpha-b}\pm h\sqrt{\alpha-c})^2,$$

where $\sqrt{\alpha-b}$, $\sqrt{\alpha-c}$ are real;

when

$$\lambda=\beta,\quad y=(b-\beta)g^2+(c-\beta)h^2\mp 2\sqrt{(b-\beta)(c-\beta)}gh$$
$$=(g\sqrt{b-\beta}\mp h\sqrt{c-\beta})^2,$$

where $\sqrt{b-\beta}$, $\sqrt{c-\beta}$ are real.

Hence from the graph we see that the equation $\phi(\lambda)=0$ has three real roots, λ_1, λ_2, λ_3, such that

$$\lambda_3<\beta<\lambda_2<\alpha<\lambda_1.$$

The above proof fails if α and β, the roots of the equation $\psi(\lambda)=0$, are equal. In that case, however, we have $b=c$, and $f=0$; and therefore the cubic becomes

$$(\lambda-b)\{(\lambda-a)(\lambda-b)-g^2-h^2\}=0,$$

the roots of which are easily seen to be all real.

145. The factors of $f(x, y, z)-\lambda(x^2+y^2+z^2)$. If λ is a root of the discriminating cubic,

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy-\lambda(x^2+y^2+z^2)$$

is the product of two factors of the form $\alpha x+\beta y+\gamma z$. Only one of the three roots leads to real values of α, β, γ. For

$$(a-\lambda)x^2+(b-\lambda)y^2+(c-\lambda)z^2+2fyz+2gzx+2hxy$$
$$\equiv\frac{1}{(b-\lambda)}\left[\{hx+(b-\lambda)y+fz\}^2+\frac{1}{\psi(\lambda)}\{z\psi(\lambda)-[hf-(b-\lambda)g]x\}^2\right],*$$

and therefore is of the form

$$\frac{1}{(b-\lambda)}\left[u^2+\frac{1}{\psi(\lambda)}v^2\right],$$

where u and v are linear functions of x, y, z, with real

---

* $ax^2+by^2+cz^2+2fyz+2gzx+2hxy$

$$\equiv\frac{1}{b}\left[(hx+by+fz)^2+(\mathrm{C}x^2-2\mathrm{G}zx+\mathrm{A}z^2)\right],$$
$$\equiv\frac{1}{b}\left[(hx+by+fz)^2+\frac{1}{\mathrm{A}}(\mathrm{A}z-\mathrm{G}x)^2\right],$$

if $\Delta=0$ and therefore $\mathrm{G}^2=\mathrm{AC}$.

coefficients. Reference to the graphs shews that the signs of $b-\lambda$ and $\psi(\lambda)$ for $\lambda=\lambda_1, \lambda_2, \lambda_3$ are as follows:

	$b-\lambda$,	$\psi(\lambda)$,
$\lambda=\lambda_1$,	$-$,	$+$,
$\lambda=\lambda_2$,	$\pm$,	$-$,
$\lambda=\lambda_3$,	$+$,	$+$.

Hence $f(x, y, z)-\lambda(x^2+y^2+z^2)$ takes the forms

$$-\mathsf{L}^2u^2-\mathsf{M}^2v^2, \quad \mathsf{L}^2u^2-\mathsf{M}^2v^2, \quad \mathsf{L}^2u^2+\mathsf{M}^2v^2,$$

according as $\lambda=\lambda_1$, λ_2, or λ_3. The factors with real coefficients correspond therefore to the mean root, λ_2. (Cf. § 93.)

146. Conditions for equal roots. *To find the conditions that the discriminating cubic should have,* (i) *two roots equal,* (ii) *three roots equal.*

The cubic is

$$\phi(\lambda)\equiv\begin{vmatrix} a-\lambda, & h, & g \\ h, & b-\lambda, & f \\ g, & f, & c-\lambda \end{vmatrix}=0.$$

Therefore, as in § 141, if λ is a root of the cubic,

$$(b-\lambda)(c-\lambda)-f^2, \quad (c-\lambda)(a-\lambda)-g^2, \quad (a-\lambda)(b-\lambda)-h^2$$

have the same sign.

(i) If λ is a repeated root, $\phi(\lambda)=0$ and

$$-\frac{d\phi}{d\lambda}\equiv(b-\lambda)(c-\lambda)-f^2$$
$$+(c-\lambda)(a-\lambda)-g^2+(a-\lambda)(b-\lambda)-h^2=0,$$

and therefore

$$\left.\begin{array}{c}(b-\lambda)(c-\lambda)=f^2, \quad (c-\lambda)(a-\lambda)=g^2, \\ (a-\lambda)(b-\lambda)=h^2,\end{array}\right\} \quad \text{........(A)}$$

and hence, (corresponding to $\mathsf{F}=\mathsf{G}=\mathsf{H}=0$), we have also

$$(a-\lambda)f=gh, \quad (b-\lambda)g=hf, \quad (c-\lambda)h=fg. \quad \text{......(B)}$$

Any one of the three sets of conditions,

$$(c-\lambda)(a-\lambda)=g^2, \quad (a-\lambda)(b-\lambda)=h^2, \quad (a-\lambda)f=gh, \quad (\text{A}')$$
$$(a-\lambda)(b-\lambda)=h^2, \quad (b-\lambda)(c-\lambda)=f^2, \quad (b-\lambda)g=hf, \quad (\text{B}')$$
$$(b-\lambda)(c-\lambda)=f^2, \quad (c-\lambda)(a-\lambda)=g^2, \quad (c-\lambda)h=fg, \quad (\text{C}')$$

is both necessary and sufficient. For if (A′) is given, substituting for $a-\lambda$ from the third equation in the first two, we obtain

$$(b-\lambda)g=hf, \quad (c-\lambda)h=fg,$$

whence

$$(b-\lambda)(c-\lambda)=f^2.$$

Therefore, if none of the three quantities f, g, h is zero, λ, the repeated root, is not equal to a, b or c, and we have

$$\lambda=a-\frac{gh}{f}=b-\frac{hf}{g}=c-\frac{fg}{h},$$

or

$$-\lambda=\frac{\mathsf{F}}{f}=\frac{\mathsf{G}}{g}=\frac{\mathsf{H}}{h}. \qquad \text{.........................(1)}$$

If f, one of the three quantities f, g, h, is zero, then, from (A), $\lambda=b$ or c. If $\lambda=b$, then $h=0$, and

$$(c-b)(a-b)=g^2.$$

If $\lambda=c$, then $g=0$, and

$$(a-c)(b-c)=h^2.$$

Therefore if one of the three quantities is zero, another must be zero, and we have

$$\left.\begin{array}{lll} \lambda=a, & g=h=0, & (b-a)(c-a)=f^2; \\ \text{or} \quad \lambda=b, & h=f=0, & (c-b)(a-b)=g^2; \\ \text{or} \quad \lambda=c, & f=g=0, & (a-c)(b-c)=h^2. \end{array}\right\} \text{.........(2)}$$

The equations (1) and (2) give the conditions for a pair of equal roots and the value of the roots in each case.

(ii) If the three roots are equal to λ, λ also satisfies the equation

$$\frac{d^2\phi}{d\lambda^2}=0, \quad \text{or} \quad (a-\lambda)+(b-\lambda)+(c-\lambda)=0.$$

But, by (A),

$$\{(a-\lambda)+(b-\lambda)+(c-\lambda)\}^2$$
$$=(a-\lambda)^2+(b-\lambda)^2+(c-\lambda)^2+2f^2+2g^2+2h^2.$$

Therefore $\lambda=a=b=c$, and $f=g=h=0$. The conicoid in this case must be a sphere.

147. The principal directions. We shall call the directions determined by the equations

$$\frac{al+hm+gn}{l}=\frac{hl+bm+fn}{m}=\frac{gl+fm+cn}{n}=\lambda,$$

or

$$\frac{\frac{\partial f}{\partial l}}{2l}=\frac{\frac{\partial f}{\partial m}}{2m}=\frac{\frac{\partial f}{\partial n}}{2n}=\lambda,$$

the **principal directions.**

If λ is a root of the discriminating cubic giving values l, m, n of the direction-cosines of a principal direction, $\lambda=f(l, m, n)$.

For
$$\lambda=\frac{\frac{\partial f}{\partial l}}{2l}=\frac{\frac{\partial f}{\partial m}}{2m}=\frac{\frac{\partial f}{\partial n}}{2n}=\frac{l\frac{\partial f}{\partial l}+m\frac{\partial f}{\partial m}+n\frac{\partial f}{\partial n}}{2(l^2+m^2+n^2)}=f(l, m, n).$$

148. *The principal directions corresponding to two distinct roots of the discriminating cubic are at right angles.*

If λ_1, λ_2 are the roots, and l_1, m_1, n_1; l_2, m_2, n_2 are the corresponding direction-cosines, then

$$2l_1\lambda_1=\frac{\partial f}{\partial l_1}, \text{ etc.}; \quad 2l_2\lambda_2=\frac{\partial f}{\partial l_2}, \text{ etc.}$$

But
$$l_1\frac{\partial f}{\partial l_2}+m_1\frac{\partial f}{\partial m_2}+n_1\frac{\partial f}{\partial n_2}\equiv l_2\frac{\partial f}{\partial l_1}+m_2\frac{\partial f}{\partial m_1}+n_2\frac{\partial f}{\partial n_1},$$

and therefore

$$(\lambda_1-\lambda_2)(l_1l_2+m_1m_2+n_1n_2)=0,$$

which proves the proposition.

149. Cases of equal roots. (i) If $\lambda_1, \lambda_2, \lambda_3$ are the roots of the discriminating cubic, and $\lambda_2=\lambda_3$, there is a definite principal direction corresponding to λ_1; but the equations

$$\frac{\frac{\partial f}{\partial l_2}}{2l_2}=\frac{\frac{\partial f}{\partial m_2}}{2m_2}=\frac{\frac{\partial f}{\partial n_2}}{2n_2}=\lambda_2 \quad \text{.....................(1)}$$

reduce to a single equation which is satisfied by the direction-cosines of any direction at right angles to the principal direction corresponding to λ_1.

Suppose that we have

$$\lambda_2=\lambda_3\neq 0, \quad \text{and} \quad \lambda_2=a-\frac{gh}{f}=b-\frac{hf}{g}=c-\frac{fg}{h}.$$

Then the equations

$$(a-\lambda_2)l_2+hm_2+gn_2=0, \quad hl_2+(b-\lambda_2)m_2+fn_2=0,$$

$$gl_2+fm_2+(c-\lambda_2)n_2=0$$

all become

$$ghl_2+hfm_2+fgn_2=0.$$

And since the sum of the roots of the cubic is $a+b+c$,

$$a-\lambda_1=-\frac{hf}{g}-\frac{fg}{h},$$

and hence the equation

$$(a-\lambda_1)l_1+hm_1+gn_1=0$$

may be written

$$l_1\left(\frac{gh}{f}+\frac{hf}{g}+\frac{fg}{h}\right)=gh\left(\frac{l_1}{f}+\frac{m_1}{g}+\frac{n_1}{h}\right).$$

The three equations corresponding to λ_1 therefore give

$$\frac{l_1}{gh}=\frac{m_1}{hf}=\frac{n_1}{fg},$$

which determine a definite principal direction. The single equation corresponding to λ_2 is the condition that the directions given by $gh:hf:fg$, $l_2:m_2:n_2$ should be at right angles.

If we have

$$\lambda_2=\lambda_3=a, \quad g=0, \quad h=0, \quad \text{and} \quad (b-a)(c-a)=f^2,$$

the equations corresponding to λ_2 and λ_1 are

$$(b-a)m_2+fn_2=0,$$

$$\frac{l_1}{0}=\frac{m_1}{b-a}=\frac{n_1}{f}.$$

If $\qquad \lambda_2=\lambda_3=0, \quad \text{then} \quad \mathsf{D}=0,$

$$bc=f^2 \quad ca=g^2, \quad ab=h^2;$$

and the equations corresponding to λ_2 and λ_1 are

$$\sqrt{a}\,l_2+\sqrt{b}\,m_2+\sqrt{c}\,n_2=0,$$

$$\frac{l_1}{\sqrt{a}}=\frac{m_1}{\sqrt{b}}=\frac{n_1}{\sqrt{c}}.$$

(ii) If the discriminating cubic has three equal roots, any direction is a principal direction. For $\lambda=a=b=c$, and $f=g=h=0$, and the equations for principal directions reduce to

$$\frac{l}{l}=\frac{m}{m}=\frac{n}{n}.$$

The reason is obvious. The surface is a sphere, and any plane through the centre bisects chords at right angles to it.

To sum up, if the discriminating cubic has distinct roots, there are three mutually perpendicular principal directions. If it has two or three repeated roots, three mutually perpendicular directions can be chosen whose direction-cosines satisfy the equations of § 147, which determine the principal directions. Therefore in all cases we can transform the equation, taking as new rectangular axes three lines through the origin whose direction-cosines satisfy the equations

$$\frac{\dfrac{\partial f}{\partial l}}{2l}=\frac{\dfrac{\partial f}{\partial m}}{2m}=\frac{\dfrac{\partial f}{\partial n}}{2n}=\lambda,$$

where λ is a root of the discriminating cubic.

150. Transformation of $f(x, y, z)$.

$f(x, y, z)$ *transforms into* $\lambda_1\xi^2+\lambda_2\eta^2+\lambda_3\zeta^2$.

Let $O\xi$, $O\eta$, $O\zeta$ have direction-cosines l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3, corresponding to the roots $\lambda_1, \lambda_2, \lambda_3$ of the cubic.

Then $x=l_1\xi+l_2\eta+l_3\zeta$, etc. $\xi=l_1x+m_1y+n_1z$, etc.

We have also

$$l_1\frac{\partial f}{\partial x}+m_1\frac{\partial f}{\partial y}+n_1\frac{\partial f}{\partial z}\equiv x\frac{\partial f}{\partial l_1}+y\frac{\partial f}{\partial m_1}+z\frac{\partial f}{\partial n_1},$$
$$=2\lambda_1(l_1x+m_1y+n_1z),$$
$$=2\lambda_1\xi.$$

And similarly,

$$l_2\frac{\partial f}{\partial x}+m_2\frac{\partial f}{\partial y}+n_2\frac{\partial f}{\partial z}=2\lambda_2\eta,$$

$$l_3\frac{\partial f}{\partial x}+m_3\frac{\partial f}{\partial y}+n_3\frac{\partial f}{\partial z}=2\lambda_3\zeta.$$

Hence, multiplying by ξ, η, ζ respectively and adding,

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=2(\lambda_1\xi^2+\lambda_2\eta^2+\lambda_3\zeta^2),$$

or
$$f(x,\ y,\ z)=\lambda_1\xi^2+\lambda_2\eta^2+\lambda_3\zeta^2.$$

Ex. 1. (i) In Ex. (i), § 140,

$14x^2+14y^2+8z^2-4yz-4zx-8xy$ transforms into $6\xi^2+12\eta^2+18\zeta^2$.

(ii) In Ex. 1, (i), § 142,

$2x^2+20y^2+18z^2-12yz+12xy$ transforms into $14\xi^2+26\eta^2$.

(iii) In Ex. (i), § 143,

$x^2+y^2+z^2-2yz+2zx-2xy$ transforms into $3\xi^2$.

Ex. 2. Prove that the conicoids

$$2ayz+2bzx+2cxy=1, \quad 2\alpha yz+2\beta zx+2\gamma xy=1$$

can be placed so as to be confocal if

$$\frac{abc}{a^2+b^2+c^2}+\frac{\alpha\beta\gamma}{\alpha^2+\beta^2+\gamma^2}=0 \text{ and } \frac{a^2b^2c^2}{(a^2+b^2+c^2)^3}+\frac{\alpha^2\beta^2\gamma^2}{(\alpha^2+\beta^2+\gamma^2)^3}=\frac{1}{27}.$$

THE CENTRE.

151. If there is a point O, such that when P is any point on a conicoid and PO is produced its own length to P′, P′ is also on the conicoid, O is a **centre** of the conicoid.

If the origin is at a centre, the coefficients of x, y, z in the equation to the conicoid are zero.

Let the equation be

$$f(x,\ y,\ z)+2ux+2vy+2wz+d=0.$$

Then if P is $(x',\ y',\ z')$, P′ is $(-x',\ -y',\ -z')$, and

$$f(x',\ y',\ z')+2ux'+2vy'+2wz'+d=0,$$
$$f(x',\ y',\ z')-2ux'-2vy'-2wz'+d=0.$$

Therefore
$$ux'+vy'+wz'=0. \quad \text{.....................}(1)$$

Hence, since equation (1) is satisfied by an infinite number of values of x', y', z' other than the coordinates of points lying in the plane

$$ux+vy+wz=0,$$

we must have
$$u=v=w=0.$$

152. *To determine the centres of the conicoid*

$$F(x, y, z)=0.$$

Let (α, β, γ) be a centre. Change the origin to (α, β, γ) and the equation becomes

$$F(x+\alpha,\ y+\beta,\ z+\gamma)=0,$$

or $$f(x, y, z)+x\frac{\partial F}{\partial \alpha}+y\frac{\partial F}{\partial \beta}+z\frac{\partial F}{\partial \gamma}+F(\alpha, \beta, \gamma)=0.$$

Therefore, since the coefficients of x, y, z are zero,

$$\frac{\partial F}{\partial \alpha}=\frac{\partial F}{\partial \beta}=\frac{\partial F}{\partial \gamma}=0.$$

Cor. The equation to any diametral plane is of the form

$$l\frac{\partial F}{\partial x}+m\frac{\partial F}{\partial y}+n\frac{\partial F}{\partial z}=0,$$

and therefore any diametral plane passes through the centre or centres.

153. The central planes. The equations

$$\frac{1}{2}\frac{\partial F}{\partial x}\equiv ax+hy+gz+u=0, \qquad \text{.................(1)}$$

$$\frac{1}{2}\frac{\partial F}{\partial y}\equiv hx+by+fz+v=0, \qquad \text{.................(2)}$$

$$\frac{1}{2}\frac{\partial F}{\partial z}\equiv gx+fy+cz+w=0 \qquad \text{.................(3)}$$

represent planes which we may call the **central planes.** Any point common to the central planes is a centre.

Multiply equations (1), (2), (3) by A, H, G respectively and add; then, since

$$aA+hH+gG=D \quad \text{and} \quad hA+bH+fG=0, \text{ etc.,}$$

$$x=\frac{Au+Hv+Gw}{-D}.$$

Similarly, $y=\dfrac{Hu+Bv+Fw}{-D}$, $z=\dfrac{Gu+Fv+Cw}{-D}$.

We have to consider the following cases, (cf. § 45):

I. $D\neq 0$, single centre at a finite distance, (ellipsoid, hyperboloid, or cone).

II.	$\mathrm{D}=0$, $\mathrm{A}u+\mathrm{H}v+\mathrm{G}w\neq 0$,	single centre at an infinite distance,	(paraboloid).
III.	$\mathrm{D}=0$, $\mathrm{A}u+\mathrm{H}v+\mathrm{G}w=0$, $\mathrm{A}\neq 0$,	line of centres at a finite distance, (central planes pass through one line and are not parallel,)	(elliptic or hyperbolic cylinder, pair of intersecting planes).
IV.	A, B, C, F, G, H, all zero, $fu\neq gv$,	line of centres at an infinite distance, (central planes parallel but not coincident,)	(parabolic cylinder).
V.	A, B, C, F, G, H, all zero, $fu=gv=wh$,	plane of centres, (central planes coincident,)	(pair of parallel planes).

154. Equation when the origin is at a centre. If the conicoid has a centre (α, β, γ) at a finite distance, and the origin is changed to it, the equation becomes

$$f(x, y, z)+\mathrm{F}(\alpha, \beta, \gamma)=0,$$

or, since
$$\alpha\frac{\partial\mathrm{F}}{\partial\alpha}+\beta\frac{\partial\mathrm{F}}{\partial\beta}+\gamma\frac{\partial\mathrm{F}}{\partial\gamma}=0,$$

$$f(x, y, z)+u\alpha+v\beta+w\gamma+d=0.$$

This becomes, on substituting the coordinates of the centre found in § 153,

$$f(x, y, z)=\frac{\mathrm{A}u^2+\mathrm{B}v^2+\mathrm{C}w^2+2\mathrm{F}vw+2\mathrm{G}wu+2\mathrm{H}uv-d\mathrm{D}}{\mathrm{D}}$$

$$=\frac{-\mathrm{S}}{\mathrm{D}}, \quad \text{where } \mathrm{S}\equiv\begin{vmatrix} a, & h, & g, & u \\ h, & b, & f, & v \\ g, & f, & c, & w \\ u, & v, & w, & d \end{vmatrix}.$$

Cor. If $\mathrm{F}(x, y, z)=0$ represents a cone, $\mathrm{S}=0$ and $\mathrm{D}\neq 0$.

Ex. 1. Find the centres of the conicoids,

(i) $14x^2+14y^2+8z^2-4yz-4zx-8xy+18x-18y+5=0$,

(ii) $3x^2+5y^2+3z^2-2yz+2zx-2xy+2x+12y+10z+20=0$,

(iii) $2x^2+20y^2+18z^2-12yz+12xy+22x+6y-2z-2=0$,

(iv) $5x^2+26y^2+10z^2+4yz+14zx+6xy-8x-18y-10z+4=0$,

(v) $x^2+y^2+z^2-2yz+2zx-2xy-2x-4y-2z+3=0$,

(vi) $x^2+y^2+z^2-2yz+2zx-2xy-2x+2y-2z-3=0$.

Ans. (i) $\left(-\frac{1}{2}, \frac{1}{2}, 0\right)$, (ii) $\left(-\frac{1}{6}, -\frac{5}{3}, -\frac{13}{6}\right)$, (iii) the central planes are parallel to the line $\frac{x}{-9}=\frac{y}{3}=\frac{z}{1}$, (iv) $\frac{x-5}{-16}=\frac{y}{1}=\frac{z+3}{11}$ is the line of centres, (v) the central planes are parallel, (vi) $x-y+z=1$ is the plane of centres.

Ex. 2. If the origin is changed to the centre, the equations (i) and (ii) become

$$14x^2+14y^2+8z^2-4yz-4zx-8xy=4,$$
$$3x^2+5y^2+3z^2-2yz+2zx-2xy=1.$$

Ex. 3. If the origin is changed to $(5, 0, -3)$, or to any point on the line of centres, the equation (iv) becomes

$$5x^2+26y^2+10z^2+4yz+14zx+6xy=1.$$

Ex. 4. Prove that the centres of conicoids that pass through the ellipses $x^2/a^2+y^2/b^2=1$, $z=0$; $x^2/a^2+z^2/c^2=1$, $y=0$ lie on the lines

$$\frac{x}{0}=\frac{y}{b}=\frac{z}{\pm c}.$$

Ex. 5. The locus of the centres of conicoids that pass through two given straight lines and two given points is a straight line.

Ex. 6. If $F(x, y, z, t)=0$ represents a cone, the coordinates of the vertex satisfy the equations

$$\frac{\partial F}{\partial x}=\frac{\partial F}{\partial y}=\frac{\partial F}{\partial z}=\frac{\partial F}{\partial t}=0.$$

Ex. 7. Through the sections of a system of confocals by one of the principal planes and by a given plane, cones are described. Prove that their vertices lie on a conic.

Ex. 8. Prove that the centres of conicoids that pass through the circle $x^2+y^2=2ax$, $z=0$, and have OZ as a generator, lie on the cylinder $x^2+y^2=ax$.

Ex. 9. A conicoid touches the axes (rectangular) at the fixed points $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$, and its section by the plane through these points is a circle. Shew that its centre lies on the line

$$\frac{x}{a^3(b^2+c^2)}=\frac{y}{b^3(c^2+a^2)}=\frac{z}{c^3(a^2+b^2)}.$$

Ex. 10. Shew that the locus of the centres of conicoids which touch the plane $z=0$ at an umbilic at the origin, touch the plane $x=a$ and pass through a fixed point on the z-axis, is a conicoid which touches the plane $z=0$ at an umbilic.

Ex. 11. Variable conicoids pass through the given conics

$$z=0,\ ax^2+by^2+2fx+d=0\ ;\quad x=0,\ cz^2+by^2+2gz+d=0\ ;$$

shew that the locus of their centres is a conic in the plane $y=0$.

Ex. 12. Find the locus of the centres of conicoids that pass through two conics which have two common points.

REDUCTION OF THE GENERAL EQUATION.

155. Case A: $\mathsf{D}\neq 0$.

There is a single centre at a finite distance, (§ 153, I.). Change the origin to it, and the equation becomes

$$f(x, y, z)+\frac{\mathsf{S}}{\mathsf{D}}=0.$$

The discriminating cubic has three non-zero roots, λ_1, λ_2, λ_3, and there are three determinate principal directions, $(\lambda_1\neq\lambda_2\neq\lambda_3)$, or three directions that can be taken as principal directions, $(\lambda_2=\lambda_3$, or $\lambda_1=\lambda_2=\lambda_3)$. The lines through the centre in these directions are the principal axes of the surface. They are the lines of intersection of the principal planes. Take these lines as coordinate axes, and the equation transforms into

$$\lambda_1x^2+\lambda_2y^2+\lambda_3z^2+\frac{\mathsf{S}}{\mathsf{D}}=0.$$

The surface is thus an ellipsoid, a hyperboloid of one sheet, a hyperboloid of two sheets, or a sphere, if $\mathsf{S}\neq 0$. If $\mathsf{S}=0$, the surface is a cone.

Ex 1. $x^2+y^2+z^2-6yz-2zx-2xy-6x-2y-2z+2=0$.
The discriminating cubic is

$$\lambda^3-3\lambda^2-8\lambda+16=0.$$

Whence $\quad \lambda=4,\ \dfrac{-1+\sqrt{17}}{2},\ \dfrac{-1-\sqrt{17}}{2};\quad 4,\ \alpha^2,\ -\beta^2,$ say.

For the centre

$$x-y-z-3=0,$$
$$-x+y-3z-1=0,$$
$$-x-3y+z-1=0.$$

These give $\quad x=1,\ y=-1,\ z=-1.$

$$\therefore\ \frac{\mathsf{S}}{\mathsf{D}}=ux+vy+wz+d=1.$$

The reduced equation is therefore $-4x^2-\alpha^2y^2+\beta^2z^2=1$, and represents a hyperboloid of two sheets.

Note. If the roots of the discriminating cubic cannot be found by inspection, their signs may be determined by a corollary of Descartes' Rule of Signs: "If the roots of $f(\lambda)=0$ are all real, the number of positive roots is equal to the number of changes of sign in $f(\lambda)$." In the above case, $f(\lambda)\equiv\lambda^3+3\lambda^2-8\lambda+16$, and there are two changes of sign, and therefore two positive roots.

Ex 2. Reduce:

$$\text{(i) } x^2+3y^2+3z^2-2yz-2x-2y+6z+3=0,$$
$$\text{(ii) } 3x^2-y^2-z^2+6yz-6x+6y-2z-2=0,$$
$$\text{(iii) } 2y^2+4zx+2x-4y+6z+5=0.$$

Ans. (i) $x^2+2y^2+4z^2=1$, (ii) $3x^2+2y^2-4z^2=4$, (iii) $x^2+y^2-z^2=0$.

156. Case B: $D=0$, $Au+Hv+Gw\neq0$.

There is a single centre at infinity, (§ 153, II.). If λ_1, λ_2, λ_3 are the roots of the discriminating cubic, $\lambda_1\neq0$, $\lambda_2\neq0$, $\lambda_3=0$, (§ 141). If l_3, m_3, n_3 are the direction-cosines of the principal direction corresponding to λ_3,

$$gl_3+fm_3+cn_3=0$$

and
$$hl_3+bm_3+fn_3=0,$$

and therefore
$$\frac{l_3}{A}=\frac{m_3}{H}=\frac{n_3}{G}=\frac{ul_3+vm_3+wn_3}{uA+vH+wG}.$$

Hence $ul_3+vm_3+wn_3\neq0$. Denote it by k.

The principal plane corresponding to λ_3 is at infinity, (§ 142).

Where the line $\dfrac{x-\alpha}{l_3}=\dfrac{y-\beta}{m_3}=\dfrac{z-\gamma}{n_3}=r$

meets the surface, we have

$$r^2f(l_3, m_3, n_3)+r\left(l_3\frac{\partial F}{\partial\alpha}+m_3\frac{\partial F}{\partial\beta}+n_3\frac{\partial F}{\partial\gamma}\right)+F(\alpha, \beta, \gamma)=0,$$

which, by means of the equations of § 147, may be written

$$\lambda_3r^2+2r\{\lambda_3(l_3\alpha+m_3\beta+n_3\gamma)+ul_3+vm_3+wn_3\}$$
$$+F(\alpha, \beta, \gamma)=0,$$

or
$$0\,.\,r^2+2kr+F(\alpha, \beta, \gamma)=0.$$

Therefore any line in the principal direction corresponding to λ_3 meets the surface in one point at a finite and one point at an infinite distance.

If the line
$$\frac{x}{l_3}=\frac{y}{m_3}=\frac{z}{n_3}$$

is normal to the tangent plane at (x', y', z'), a point of the surface,

$$\frac{\dfrac{\partial F}{\partial x'}}{l_3}=\frac{\dfrac{\partial F}{\partial y'}}{m_3}=\frac{\dfrac{\partial F}{\partial z'}}{n_3}=\frac{l_3\dfrac{\partial F}{\partial x'}+m_3\dfrac{\partial F}{\partial y'}+n_3\dfrac{\partial F}{\partial z'}}{l_3^2+m_3^2+n_3^2},$$

$$=2(ul_3+vm_3+wn_3)=2k.$$

Hence such points as (x', y', z') lie on the three planes

$$p_1\equiv ax+hy+gz+u-kl_3=0, \quad \ldots\ldots\ldots\ldots\ldots(1)$$

$$p_2\equiv hx+by+fz+v-km_3=0, \quad \ldots\ldots\ldots\ldots\ldots(2)$$

$$p_3\equiv gx+fy+cz+w-kn_3=0. \quad \ldots\ldots\ldots\ldots\ldots(3)$$

But
$$l_3p_1+m_3p_2+n_3p_3\equiv 0,$$

therefore the three planes pass through a line which is parallel to $\dfrac{x}{l_3}=\dfrac{y}{m_3}=\dfrac{z}{n_3}$.

Therefore there is only one point on the surface at which the normal is parallel to the line $\dfrac{x}{l_3}=\dfrac{y}{m_3}=\dfrac{z}{n_3}$. That point is the **vertex** of the surface. Its coordinates are given by the equations (1), (2), (3), (which are equivalent to two independent equations,) and the equation $F(x, y, z)=0$.

But $$F(x, y, z)\equiv \tfrac{1}{2}x\frac{\partial F}{\partial x}+\tfrac{1}{2}y\frac{\partial F}{\partial y}+\tfrac{1}{2}z\frac{\partial F}{\partial z}+ux+vy+wz+d,$$

$$\equiv k(l_3x+m_3y+n_3z)+ux+vy+wz+d.$$

Hence any two of the equations (1), (2), (3) and the equation

$$k(l_3x+m_3y+n_3z)+ux+vy+wz+d=0$$

determine the vertex.

To reduce the given equation, change the origin to the vertex, (x', y', z'). The equation becomes

$$f(x, y, z)+x\frac{\partial F}{\partial x'}+y\frac{\partial F}{\partial y'}+z\frac{\partial F}{\partial z'}+F(x', y', z')=0$$

or
$$f(x, y, z)+2k(l_3x+m_3y+n_3z)=0.$$

Take the three lines through the vertex in the principal directions as coordinate axes, so that $O\zeta$ has direction-cosines l_3, m_3, n_3, and the equation transforms into

$$\lambda_1\xi^2+\lambda_2\eta^2+2k\zeta=0.$$

The surface is therefore a paraboloid.

Since, from (1), (2), (3),

$$l_1\frac{\partial F}{\partial x'}+m_1\frac{\partial F}{\partial y'}+n_1\frac{\partial F}{\partial z'}=0,\quad l_2\frac{\partial F}{\partial x'}+m_2\frac{\partial F}{\partial y'}+n_2\frac{\partial F}{\partial z'}=0,$$

the principal planes corresponding to λ_1 and λ_2 pass through the vertex. The new coordinate planes are therefore the two principal planes at a finite distance and the tangent plane at the vertex.

Ex. 1. $2x^2+2y^2+z^2+2yz-2zx-4xy+x+y=0.$

The discriminating cubic is

$$\lambda^3-5\lambda^2+2\lambda=0.$$

Whence $\lambda=\dfrac{5+\sqrt{17}}{2},\ \dfrac{5-\sqrt{17}}{2},\ 0;\quad \alpha^2,\ \beta^2,\ 0,\quad$ say.

For the principal direction corresponding to λ_3,

$$2l_3-2m_3-n_3=0,$$
$$-l_3+m_3+n_3=0.$$

Therefore $$\frac{l_3}{1}=\frac{m_3}{1}=\frac{n_3}{0}=\frac{1}{\sqrt{2}},$$

and $$k=ul_3+vm_3+wn_3=\frac{1}{\sqrt{2}}.$$

The reduced equation is therefore $\alpha^2x^2+\beta^2y^2+\dfrac{2}{\sqrt{2}}z=0.$

The equations for the vertex give $x=y=z=0$, and the axis is $x=y,\ z=0$.

Ex. 2. The following equations represent paraboloids. Find the reduced equations, the coordinates of the vertex, and the equations to the axis.

(i) $4y^2+4z^2+4yz-2x-14y-22z+33=0,$

(ii) $4x^2-y^2-z^2+2yz-8x-4y+8z-2=0.$

Ans. (i) $y^2+3z^2=x$; $(1,\ 1/2,\ 5/2)$; $\dfrac{x-1}{1}=\dfrac{2y-1}{0}=\dfrac{2z-5}{0}$;

(ii) $2x^2-y^2+\sqrt{2}z=0$; $(1,\ -9/4,\ 3/4)$; $\dfrac{x-1}{0}=\dfrac{4y+9}{1}=\dfrac{4z-3}{1}.$

157. Case C: $D=0,\ Au+Hv+Gw=0,\ A\neq 0.$

There is a line of centres at a finite distance, (§ 153, III.).

The discriminating cubic has one zero-root, λ_3, giving, (as in § 156),

$$\frac{l_3}{A}=\frac{m_3}{H}=\frac{n_3}{G},$$

or, since $GH=AF$, $$\frac{l_3}{1/F}=\frac{m_3}{1/G}=\frac{n_3}{1/H}.$$

In this case $ul_3+vm_3+wn_3=0$, and the principal plane corresponding to λ_3 is indeterminate, (§ 142). It may be any plane at right angles to $\mathsf{F}x=\mathsf{G}y=\mathsf{H}z$.

The line of centres has equations

$$ax+hy+gz+u=0,$$
$$hx+by+fz+v=0,$$

which may be written

$$\frac{x-\dfrac{uf}{\mathsf{F}}}{1/\mathsf{F}}=\frac{y-\dfrac{vg}{\mathsf{G}}}{1/\mathsf{G}}=\frac{z-\dfrac{wh}{\mathsf{H}}}{1/\mathsf{H}}.$$

Hence l_3, m_3, n_3 are the direction-cosines of the line of centres.

Any point on the line of centres has coordinates

$$\frac{uf+r}{\mathsf{F}},\ \frac{vg+r}{\mathsf{G}},\ \frac{wh+r}{\mathsf{H}}.$$

If we change the origin to it, $\mathsf{F}(x, y, z)=0$ becomes

$$f(x, y, z)+\frac{u(uf+r)}{\mathsf{F}}+\frac{v(vg+r)}{\mathsf{G}}+\frac{w(wh+r)}{\mathsf{H}}+d=0,$$

or, since $\qquad ul_3+vm_3+wn_3=0,$

$$f(x, y, z)+\frac{u^2f}{\mathsf{F}}+\frac{v^2g}{\mathsf{G}}+\frac{w^2h}{\mathsf{H}}+d=0.$$

Transform now to axes through the centre chosen whose directions are the principal directions, and the equation further reduces to $\quad \lambda_1x^2+\lambda_2y^2+d'=0,$

where $$d'\equiv\frac{u^2f}{\mathsf{F}}+\frac{v^2g}{\mathsf{G}}+\frac{w^2h}{\mathsf{H}}+d.$$

If $d'\neq 0$, the surface is an elliptic or hyperbolic cylinder; if $d'=0$, it is a pair of intersecting planes.

Ex. 1. $x^2+6y^2-z^2-yz+5xy+2x+5y=0.$

The discriminating cubic is

$$\lambda^3-6\lambda^2-\frac{15}{2}\lambda=0.$$

Whence $\qquad \lambda=3\pm\sqrt{\frac{33}{2}},\ 0;\ \ \alpha^2,\ -\beta^2,\ 0,\ \text{say.}$

Corresponding to $\lambda_3=0$, we have

$$2l_3+5m_3=0,\quad 5l_3+12m_3-n_3=0,$$

and therefore $$\frac{l_3}{5}=\frac{m_3}{-2}=\frac{n_3}{1},$$

and $$ul_3+vm_3+wn_3=0.$$

Hence there is a line of centres, given by $2x+5y+2=0$, $y+2z=0$. A point on this line is $(-1, 0, 0)$. Change the origin to it, and the given equation becomes

$$x^2+6y^2-z^2-yz+5xy=1,$$

which reduces to $\alpha^2x^2-\beta^2y^2=1$.

Ex. 2. What surfaces are represented by

(i) $2y^2-2yz+2zx-2xy-x-2y+3z-2=0$,

(ii) $26x^2+20y^2+10z^2-4yz-16zx-36xy+52x-36y-16z+25=0$?

Ans. (i) $6x^2-2y^2=1$, axis $2x+3=2y=2z-1$;
(ii) $14x^2+42y^2=1$, axis $x=y-1=z-1$.

Ex. 3. Prove that the equation

$$5x^2-4y^2+5z^2+4yz-14zx+4xy+16x+16y-32z+8=0$$

represents a pair of planes which pass through the line $x+3=y=z+1$ and are inclined at an angle $2\tan^{-1}\sqrt{2}$.

158. Case D: $A=B=C=F=G=H=0$, $uf \neq vg$.

There is a line of centres at infinity, (§ 153, IV.). If $\lambda_1, \lambda_2, \lambda_3$ are the roots of the discriminating cubic,

$$\lambda_1=a+b+c, \quad \lambda_2=\lambda_3=0.$$

If l_1, m_1, n_1 are the direction-cosines of the principal direction corresponding to λ_1, since $f^2=bc$, $g^2=ca$, and $h^2=ab$, we have

$$\frac{al_1+\sqrt{ab}m_1+\sqrt{ac}n_1}{l_1}=\frac{\sqrt{ab}l_1+bm_1+\sqrt{bc}n_1}{m_1}$$
$$=\frac{\sqrt{ca}l_1+\sqrt{bc}m_1+cn_1}{n_1},$$

whence $$\frac{l_1}{\sqrt{a}}=\frac{m_1}{\sqrt{b}}=\frac{n_1}{\sqrt{c}}.$$

And since $uf-vg \neq 0$, $u\sqrt{b}-v\sqrt{a} \neq 0$, or $um_1-vl_1 \neq 0$.

Let $O\xi$, $O\eta$, $O\zeta$ be a set of rectangular axes whose direction-cosines are l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3. Then l_2, m_2, n_2; l_3, m_3, n_3 satisfy the equations for principal directions, (§ 149). The equation to the surface referred to $O\xi$, $O\eta$, $O\zeta$ is therefore

$$\lambda_1x^2+2x(ul_1+vm_1+wn_1)+2y(ul_2+vm_2+wn_2)$$
$$+2z(ul_3+vm_3+wn_3)+d=0.$$

Now $$l_1l_2+m_1m_2+n_1n_2=0,$$
and we can choose l_2, m_2, n_2 to satisfy also
$$ul_2+vm_2+wn_2=0.$$
Then $$n_2(ul_3+vm_3+wn_3)=u(n_2l_3-n_3l_2)-v(m_2n_3-m_3n_2),$$
$$=um_1-vl_1 \neq 0.$$
Therefore, if $ul_3+vm_3+wn_3$ is denoted by w_1,
$$w_1{}^2=\Sigma(um_1-vl_1)^2$$
$$=\frac{(v\sqrt{c}-w\sqrt{b})^2+(w\sqrt{a}-u\sqrt{c})^2+(u\sqrt{b}-v\sqrt{a})^2}{a+b+c}.$$

Writing u_1 for $ul_1+vm_1+wn_1$, the equation to the surface becomes
$$\lambda_1x^2+2u_1x+2w_1z+d=0,$$
or $$\left(x+\frac{u_1}{\lambda_1}\right)^2+\frac{2w_1}{\lambda_1}\left(z+\frac{d}{2w_1}-\frac{u_1{}^2}{2w_1\lambda_1}\right)=0,$$
which may be reduced, by change of origin, to
$$x^2+\frac{2w_1}{\lambda_1}z=0.$$

The surface is therefore a parabolic cylinder.

The latus rectum of a normal section
$$=\frac{2w_1}{\lambda_1}=\frac{2\{(v\sqrt{c}-w\sqrt{b})^2+(w\sqrt{a}-u\sqrt{c})^2+(u\sqrt{b}-v\sqrt{a})^2\}^{\frac{1}{2}}}{(a+b+c)^{\frac{3}{2}}}.$$

159. Case E: $\mathsf{A}=\mathsf{B}=\mathsf{C}=\mathsf{F}=\mathsf{G}=\mathsf{H}=0$, $uf=vg=wh$.

There is a plane of centres, (§ 153, V.).

As in Case D, $\lambda_1=a+b+c$, $\lambda_2=\lambda_3=0$, and
$$\frac{l_1}{\sqrt{a}}=\frac{m_1}{\sqrt{b}}=\frac{n_1}{\sqrt{c}}.$$

But since $uf=vg=wh$,
$$\frac{u}{\sqrt{a}}=\frac{v}{\sqrt{b}}=\frac{w}{\sqrt{c}},$$
and therefore
$$ul_2+vm_2+wn_2=ul_3+vm_3+wn_3=0.$$

The equation to the surface therefore reduces to
$$\lambda_1x^2+2u_1x+d=0,$$
and the surface is a pair of parallel planes.

160. Reduction when terms of second degree are a perfect square. The following method of reduction is applicable to Cases D and E, and is the most suitable method when the coefficients in the given equation are numerical.

Since $\mathsf{A}=\mathsf{B}=\mathsf{C}=0$, $f(x, y, z)$ is a perfect square.

Hence

$$\mathsf{F}(x, y, z) \equiv (\sqrt{a}x+\sqrt{b}y+\sqrt{c}z)^2+2ux+2vy+2wz+d.$$

If, (Case E), $\dfrac{u}{\sqrt{a}}=\dfrac{v}{\sqrt{b}}=\dfrac{w}{\sqrt{c}}=k$, the equation becomes

$$(\sqrt{a}x+\sqrt{b}y+\sqrt{c}z)^2+2k(\sqrt{a}x+\sqrt{b}y+\sqrt{c}z)+d=0,$$

and represents a pair of parallel planes.

But if, (Case D), $\dfrac{u}{\sqrt{a}}\neq\dfrac{v}{\sqrt{b}}$, the equation may be written

$$(\sqrt{a}x+\sqrt{b}y+\sqrt{c}z+\lambda)^2$$
$$=2x(\lambda\sqrt{a}-u)+2y(\lambda\sqrt{b}-v)+2z(\lambda\sqrt{c}-w)+\lambda^2-d.$$

Now choose λ so that the planes

$$\mathsf{U}\equiv\sqrt{a}x+\sqrt{b}y+\sqrt{c}z+\lambda=0,$$

$$\mathsf{V}\equiv 2x(\lambda\sqrt{a}-u)+2y(\lambda\sqrt{b}-v)+2z(\lambda\sqrt{c}-w)+\lambda^2-d=0$$

are at right angles. This requires that

$$\lambda=\frac{u\sqrt{a}+v\sqrt{b}+w\sqrt{c}}{a+b+c}.$$

Then take rectangular axes with $\mathsf{U}=0$, $\mathsf{V}=0$ as new coordinate planes $\xi=0$, $\eta=0$, so that

$$\xi=\frac{\mathsf{U}}{\sqrt{a+b+c}} \quad\text{and}\quad \eta=\frac{\mathsf{V}}{2\sqrt{\Sigma(\lambda\sqrt{a}-u)^2}},$$

and the equation reduces to

$$\xi^2(a+b+c)=2\sqrt{\Sigma(\lambda\sqrt{a}-u)^2}\,\eta.$$

But, by Lagrange's identity,

$$(a+b+c)\{\Sigma(\lambda\sqrt{a}-u)^2\}-\{\Sigma\sqrt{a}(\lambda\sqrt{a}-u)\}^2$$
$$\equiv\Sigma(v\sqrt{c}-w\sqrt{b})^2;$$

therefore the reduced equation may be written

$$\xi^2 = \frac{2\{\Sigma(v\sqrt{c} - w\sqrt{b})^2\}^{\frac{1}{2}}}{(a+b+c)^{\frac{3}{2}}}\eta.$$

Ex. Reduce the equations

(i) $x^2+4y^2+z^2-4yz+2zx-4xy-2x+4y-2z-3=0$,

(ii) $9x^2+4y^2+4z^2+8yz+12zx+12xy+4x+y+10z+1=0$.

Ans. (i) $6x^2-2\sqrt{6}x-3=0$, (ii) $x^2=\frac{7}{17}y$.

161. Summary of the various cases. In the reduction of the general equation of the second degree with numerical coefficients the following order of procedure is generally the most convenient:

If the terms of second degree form a perfect square, proceed as in § 160.

If the terms of second degree do not form a perfect square, solve the discriminating cubic.

If the three roots are different from zero, find the centre (α, β, γ) from the equations $\frac{\partial F}{\partial \alpha}=\frac{\partial F}{\partial \beta}=\frac{\partial F}{\partial \gamma}=0$, and the reduced equation is

$$\lambda_1 x^2+\lambda_2 y^2+\lambda_3 z^2+u\alpha+v\beta+w\gamma+d=0.$$

If one root, λ_3, is zero, find l_3, m_3, n_3, from two of the equations $\frac{\partial f}{\partial l_3}=\frac{\partial f}{\partial m_3}=\frac{\partial f}{\partial n_3}=0$. Evaluate $ul_3+vm_3+wn_3 \equiv k$. If $k \neq 0$, the reduced equation is

$$\lambda_1 x^2+\lambda_2 y^2+2kz=0.$$

If $k=0$, there is a line of centres, given by any two of the equations $\frac{\partial F}{\partial x}=\frac{\partial F}{\partial y}=\frac{\partial F}{\partial z}=0$. Choose (α, β, γ) any point on it as centre, and the reduced equation is

$$\lambda_1 x^2+\lambda_2 y^2+u\alpha+v\beta+w\gamma+d=0.$$

Ex. Reduce the following equations:

(i) $3x^2+5y^2+3z^2+2yz+2zx+2xy-4x-8z+5=0$,

(ii) $2x^2+20y^2+18z^2-12yz+12xy+22x+6y-2z-2=0$,

(iii) $3x^2-24y^2+8z^2+16yz-10zx-14xy+22y+2z-4=0$,

(iv) $36x^2+4y^2+z^2-4yz-12zx+24xy+4x+16y-26z-3=0$,

(v) $3x^2+7y^2+3z^2+10yz-2zx+10xy+4x-12y-4z+1=0$,

(vi) $6y^2-18yz-6zx+2xy-9x+5y-5z+2=0$,

(vii) $5x^2+26y^2+10z^2+4yz+14zx+6xy-8x-18y-10z+4=0$,

(viii) $4x^2+9y^2+36z^2-36yz+24zx-12xy-10x+15y-30z+6=0$,

(ix) $11y^2+14yz+8zx+14xy-6x-16y+2z-2=0$,

(x) $2x^2-7y^2+2z^2-10yz-8zx-10xy+6x+12y-6z+5=0$.

Ans. (i) $3x^2+2y^2+6z^2=1$, (ii) $14x^2+26y^2=2\sqrt{91}z$,
(iii) $14x^2-27y^2=1$, (iv) $41x^2=28y$,
(v) $3x^2-4y^2-12z^2=1$, (vi) $14x^2-26y^2=2\sqrt{91}z$,
(vii) $14x^2+27y^2=1$, (viii) $49x^2-35x+6=0$,
(ix) $3x^2+4y^2-18z^2=1$, (x) $x^2+2y^2-4z^2=0$.

162. Conicoids of revolution. If two of the roots of the discriminating cubic are equal and not zero, the equation $\mathsf{F}(x, y, z)=0$ reduces to

$$\lambda_1(x^2+y^2)+\lambda_3 z^2+\frac{\mathsf{S}}{\mathsf{D}}=0, \quad\text{.....(i)}$$

or
$$\lambda_1(x^2+y^2)+2kz=0, \quad\text{.....(ii)}$$

or
$$\lambda_1(x^2+y^2)+d'=0. \quad\text{.....(iii)}$$

The surface is therefore, (i), an ellipsoid, hyperboloid, or cone of revolution, (ii), a paraboloid of revolution, or (iii), a right circular cylinder. These are, if we exclude the sphere, the only conicoids of revolution, and therefore the conditions that $\mathsf{F}(x, y, z)=0$ should represent a surface of revolution are the conditions that the cubic should have a repeated root different from zero, viz., (§ 146),

$$\lambda_1=a-\frac{gh}{f}=b-\frac{hf}{g}=c-\frac{fg}{h}; \quad\text{.....(1)}$$

or
$$\lambda_1=a,\ (b-a)(c-a)=f^2,\ g=0,\ h=0; \quad\text{.....(2)}$$

or
$$\lambda_1=b,\ (c-b)(a-b)=g^2,\ h=0,\ f=0; \quad\text{.....(3)}$$

or
$$\lambda_1=c,\ (a-c)(b-c)=h^2,\ f=0,\ g=0. \quad\text{.....(4)}$$

If equations (1) are satisfied,

$$F(x, y, z) \equiv \lambda_1(x^2+y^2+z^2)+2ux+2vy+2wz+d + fgh\left(\frac{x}{f}+\frac{y}{g}+\frac{z}{h}\right)^2.$$

And therefore any plane parallel to the plane $\frac{x}{f}+\frac{y}{g}+\frac{z}{h}=0$ cuts the surface in a circle. The axis of the surface is the line through the centres of the circular sections, that is, the perpendicular from the centre of the sphere

$$\lambda_1(x^2+y^2+z^2)+2ux+2vy+2wz+d=0$$

to the planes of the sections. Its equations are therefore

$$f\left(x+\frac{u}{\lambda_1}\right)=g\left(y+\frac{v}{\lambda_1}\right)=h\left(z+\frac{w}{\lambda_1}\right).$$

Similarly, if equations (2) are satisfied, the equations to the axis are

$$\frac{x+u/a}{0}=\frac{y+v/a}{\sqrt{b-a}}=\frac{z+w/a}{\sqrt{c-a}}.$$

Ex. 1. Find the right circular cylinders that circumscribe the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1.$$

The enveloping cylinder whose generators are parallel to the line $x/l=y/m=z/n$ has equation

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)-\left(\frac{lx}{a^2}+\frac{my}{b^2}+\frac{nz}{c^2}\right)^2=0.$$

Conditions (1) give

$$\frac{1}{a^2}\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)=\frac{1}{b^2}\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)=\frac{1}{c^2}\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right),$$

which can only be satisfied if $a=b=c$, or $l=m=n=0$. (If $a=b=c$, the ellipsoid becomes a sphere, and any enveloping cylinder is a right cylinder.) Using conditions (2), (3), (4), we obtain

$$l=0, \quad \{m^2(a^2-c^2)+n^2(a^2-b^2)\}\{m^2c^2+n^2b^2\}=0,$$

or $$m=0, \quad \{n^2(b^2-a^2)+l^2(b^2-c^2)\}\{n^2a^2+l^2c^2\}=0,$$

or $$n=0, \quad \{l^2(c^2-b^2)+m^2(c^2-a^2)\}\{l^2b^2+m^2a^2\}=0.$$

If $a>b>c$, the second only of these equations gives real values for the direction-cosines of a generator, viz.,

$$\frac{l}{\sqrt{a^2-b^2}}=\frac{m}{0}=\frac{n}{\pm\sqrt{b^2-c^2}}.$$

If λ is the repeated root of the discriminating cubic,

$$\lambda=\frac{1}{b^2}\left(\frac{l^2}{a^2}+\frac{n^2}{c^2}\right).$$

The reduced equation to the cylinder is

$$\lambda(x^2+y^2)=\frac{l^2}{a^2}+\frac{n^2}{c^2},$$

or

$$x^2+y^2=b^2.$$

Ex. 2. Find the right circular cylinders that can be inscribed in the hyperboloid

$$x^2+2y^2-3z^2=1.$$

Ans. $4(x^2+2y^2-3z^2-1)+(x\pm\sqrt{15}z)^2=0.$

Ex. 3. Prove that

$$x^2+y^2+z^2-yz-zx-xy-3x-6y-9z+21=0$$

represents a paraboloid of revolution, and find the coordinates of the focus. *Ans.* (1, 2, 3).

Ex. 4. Find the locus of the vertices of the cones of revolution that pass through the ellipse

$$x^2/a^2+y^2/b^2=1, \quad z=0.$$

Ans. $x=0,\ \dfrac{y^2}{a^2-b^2}+\dfrac{z^2}{a^2}=-1$; $y=0,\ \dfrac{x^2}{a^2-b^2}-\dfrac{z^2}{b^2}=1.$

Ex. 5. The locus of the vertices of the right circular cones that circumscribe an ellipsoid consists of the focal conics.

Ex. 6. If $f(x, y, z)=0$ represents a right cone of semi-vertical angle α,

$$\frac{gh}{f}-a=\frac{hf}{g}-b=\frac{fg}{h}-c=\left(\frac{gh}{f}+\frac{hf}{g}+\frac{fg}{h}\right)\cos^2\alpha.$$

Ex. 7. If $f(x, y, z)=1$ represents an ellipsoid formed by the revolution of an ellipse about its major axis, the eccentricity of the generating ellipse is given by

$$\frac{a+b+c}{3-e^2}=a-\frac{gh}{f}.$$

Ex. 8. If the axes are oblique, $F(x, y, z)=0$ represents a conicoid of revolution if

$$f(x, y, z)-k(x^2+y^2+z^2+2yz\cos\lambda+2zx\cos\mu+2xy\cos\nu)$$

is a perfect square.

Hence shew that the four cones of revolution that pass through the coordinate axes are given by $ayz+bzx+cxy=0$, where

$$\frac{a}{\sin^2\frac{\lambda}{2}}=\frac{b}{\sin^2\frac{\mu}{2}}=\frac{c}{\sin^2\frac{\nu}{2}}, \quad\text{or}\quad \frac{-a}{\sin^2\frac{\lambda}{2}}=\frac{b}{\cos^2\frac{\mu}{2}}=\frac{c}{\cos^2\frac{\nu}{2}},$$

or

$$\frac{a}{\cos^2\frac{\lambda}{2}}=\frac{-b}{\sin^2\frac{\mu}{2}}=\frac{c}{\cos^2\frac{\nu}{2}}, \quad\text{or}\quad \frac{a}{\cos^2\frac{\lambda}{2}}=\frac{b}{\cos^2\frac{\mu}{2}}=\frac{-c}{\sin^2\frac{\nu}{2}}.$$

Ex. 9. Find the equations to the right circular cones that touch the (rectangular) coordinate planes.

Ans. $x^2+y^2+z^2\pm2yz\pm2zx\pm2xy=0$, (one or three of the negative signs being taken).

INVARIANTS.

163. *If the equation to a conicoid* $\mathsf{F}(x, y, z)=0$ *is transformed by any change of rectangular axes, the expressions*

$$a+b+c, \quad \mathsf{A}+\mathsf{B}+\mathsf{C}, \quad \mathsf{D}, \quad \mathsf{S}$$

remain unaltered in value.

If the origin only is changed, $f(x, y, z)$ is unaffected, and therefore $a+b+c$, $\mathsf{A}+\mathsf{B}+\mathsf{C}$, and D are unaltered.

If now the coordinate axes be turned about the origin so that $f(x, y, z)$ is transformed into

$$f_1(x, y, z) \equiv a_1x^2+b_1y^2+c_1z^2+2f_1yz+2g_1zx+2h_1xy,$$

then $$f(x, y, z)-\lambda(x^2+y^2+z^2) \quad \text{becomes}$$

$$f_1(x, y, z)-\lambda(x^2+y^2+z^2).$$

If $$f(x, y, z)-\lambda(x^2+y^2+z^2)=0$$

represents two planes,

$$f_1(x, y, z)-\lambda(x^2+y^2+z^2)=0$$

will also represent the planes. And λ is the same quantity in both equations, therefore the equations

$$\begin{vmatrix} a-\lambda, & h, & g \\ h, & b-\lambda, & f \\ g, & f, & c-\lambda \end{vmatrix}=0, \quad \begin{vmatrix} a_1-\lambda, & h_1, & g_1 \\ h_1, & b_1-\lambda, & f_1 \\ g_1, & f_1, & c_1-\lambda \end{vmatrix}=0$$

have the same roots. In each the coefficient of λ^3 is unity, and therefore

$$a+b+c=a_1+b_1+c_1,$$

$$\mathsf{A}+\mathsf{B}+\mathsf{C}=\mathsf{A}_1+\mathsf{B}_1+\mathsf{C}_1,$$

$$\mathsf{D}=\mathsf{D}_1.$$

Again if the origin is changed to (α, β, γ), $\mathsf{F}(x, y, z)=0$ becomes

$$f(x, y, z)+x\frac{\partial \mathsf{F}}{\partial \alpha}+y\frac{\partial \mathsf{F}}{\partial \beta}+z\frac{\partial \mathsf{F}}{\partial \gamma}+\mathsf{F}(\alpha, \beta, \gamma)=0.$$

Hence the new value of S,

$$\mathsf{S}' \equiv \begin{vmatrix} a, & h, & g, & \frac{1}{2}\frac{\partial \mathsf{F}}{\partial \alpha} \\ h, & b, & f, & \frac{1}{2}\frac{\partial \mathsf{F}}{\partial \beta} \\ g, & f, & c, & \frac{1}{2}\frac{\partial \mathsf{F}}{\partial \gamma} \\ \frac{1}{2}\frac{\partial \mathsf{F}}{\partial \alpha}, & \frac{1}{2}\frac{\partial \mathsf{F}}{\partial \beta}, & \frac{1}{2}\frac{\partial \mathsf{F}}{\partial \gamma}, & \mathsf{F}(\alpha, \beta, \gamma) \end{vmatrix}.$$

Multiply the numbers in the first three columns by α, β, γ respectively, and subtract the sum from the numbers in the fourth column; then apply the same process to the rows, and

$$\mathsf{S}' = \begin{vmatrix} a, & h, & g, & u \\ h, & b, & f, & v \\ g, & f, & c, & w \\ \frac{1}{2}\frac{\partial \mathsf{F}}{\partial \alpha}, & \frac{1}{2}\frac{\partial \mathsf{F}}{\partial \beta}, & \frac{1}{2}\frac{\partial \mathsf{F}}{\partial \gamma}, & u\alpha + v\beta + w\gamma + d \end{vmatrix}$$

$$= \begin{vmatrix} a, & h, & g, & u \\ h, & b, & f, & v \\ g, & f, & c, & w \\ u, & v, & w, & d \end{vmatrix} = \mathsf{S}.$$

Therefore S is unaltered by change of origin.

If the axes be now turned about the origin so that

$$\mathsf{F}(x, y, z) \equiv f(x, y, z) + 2ux + 2vy + 2wz + d$$

becomes

$$\mathsf{F}_1(x, y, z) \equiv f_1(x, y, z) + 2u_1x + 2v_1y + 2w_1z + d,$$

then $$\mathsf{F}(x, y, z) - \lambda(x^2 + y^2 + z^2 + 1)$$

transforms into

$$\mathsf{F}_1(x, y, z) - \lambda(x^2 + y^2 + z^2 + 1).$$

If $\mathsf{F}(x, y, z) - \lambda(x^2 + y^2 + z^2 + 1) = 0$ represents a cone

$$\mathsf{F}_1(x, y, z) - \lambda(x^2 + y^2 + z^2 + 1) = 0$$

will also represent the cone. And λ being the same quantity in both equations, the equations

$$\begin{vmatrix} a-\lambda, & h, & g, & u \\ h, & b-\lambda, & f, & v \\ g, & f, & c-\lambda, & w \\ u, & v, & w, & d-\lambda \end{vmatrix}=0, \quad \begin{vmatrix} a_1-\lambda, & h_1, & g_1, & u_1 \\ h_1, & b_1-\lambda, & f_1, & v_1 \\ g_1, & f_1, & c_1-\lambda, & w_1 \\ u_1, & v_1, & w_1, & d-\lambda \end{vmatrix}=0$$

have the same roots. In each equation the coefficient of λ^4 is unity, and therefore the constant terms are equal, *i.e.* $\mathsf{S}=\mathsf{S}'$. Hence S is invariant for any change of rectangular axes.

Ex. 1. If $f(x, y, z)$ transforms into $\alpha x^2+\beta y^2+\gamma z^2$, prove that α, β, γ are the roots of the discriminating cubic.

Ex. 2. If the origin is unaltered,

$$\mathsf{A}u^2+\mathsf{B}v^2+\mathsf{C}w^2+2\mathsf{F}vw+2\mathsf{G}wu+2\mathsf{H}uv$$

is invariant.

Ex. 3. If
$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy,$$
$$a_1x^2+b_1y^2+c_1z^2+2f_1yz+2g_1zx+2h_1xy$$

are simultaneously transformed,

$$a_1\mathsf{A}+b_1\mathsf{B}+c_1\mathsf{C}+2f_1\mathsf{F}+2g_1\mathsf{G}+2h_1\mathsf{H}$$

remains unaltered.

Ex. 4. If any set of rectangular axes through a fixed origin O meets a given conicoid in P, P′; Q, Q′; R, R′, prove that

$$\text{(i)}\ \frac{\mathrm{PP'}^2}{\mathrm{OP}^2.\mathrm{OP'}^2}+\frac{\mathrm{QQ'}^2}{\mathrm{OQ}^2.\mathrm{OQ'}^2}+\frac{\mathrm{RR'}^2}{\mathrm{OR}^2.\mathrm{OR'}^2},$$

$$\text{(ii)}\ \frac{1}{\mathrm{OP}.\mathrm{OP'}}+\frac{1}{\mathrm{OQ}.\mathrm{OQ'}}+\frac{1}{\mathrm{OR}.\mathrm{OR'}}$$

are constant.

Ex. 5. Shew by means of invariant expressions that the squares of the principal axes of a normal section of the cylinder which envelopes the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, and whose generators are parallel to the line $x/l=y/m=z/n$, are given by

$$\frac{l^2}{a^2-r^2}+\frac{m^2}{b^2-r^2}+\frac{n^2}{c^2-r^2}=0.$$

Examples VIII.

1. Prove that

$$5x^2+5y^2+8z^2+8yz+8zx-2xy+12x-12y+6=0$$

represents a cylinder whose cross-section is an ellipse of eccentricity $1/\sqrt{2}$, and find the equations to the axis.

2. Find the eccentricity of a section of the surface

$$x^2+y^2+z^2-\mu(yz+zx+xy)=1$$

by a plane through the line $x=y=z$.

3. What is the nature of the surface given by

$$f(x, y, z)=1 \text{ if } a-\frac{gh}{f}=b-\frac{hf}{g}=c-\frac{fg}{h}=0\,?$$

4. Prove that the cylinder and real cone through the curve of intersection of the conicoids

$$x^2+az^2=2cy, \quad y^2+bz^2=2cx$$

are given by

$$b(x^2-2cy)-a(y^2-2cx)=0, \quad x^2-y^2+(a-b)z^2+2c(x-y)=0.$$

5. Prove that three cones can be drawn through the curve of intersection of the conicoids

$$x^2+cz^2+2by+a^2=0, \quad y^2+dz^2+2ax+b^2=0,$$

and that their vertices form a right-angled triangle.

6. Prove that

$$(ax^2+by^2+cz^2-1)\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)=(lx+my+nz-1)^2$$

represents a paraboloid touching the surface $ax^2+by^2+cz^2=1$ at its points of section by the plane $lx+my+nz=1$. Prove also that its axis is parallel to the line $\frac{ax}{l}=\frac{by}{m}=\frac{cz}{n}$.

7. Shew that the conicoids

$$(a_1x+b_1y+c_1z)^2+(a_2x+b_2y+c_2z)^2+(a_3x+b_3y+c_3z)^2=1,$$
$$(a_1x+a_2y+a_3z)^2+(b_1x+b_2y+b_3z)^2+(c_1x+c_2y+c_3z)^2=1$$

are equal in all respects.

8. Prove that if $a^3+b^3+c^3=3abc$,

$$ax^2+by^2+cz^2+2ayz+2bzx+2cxy+2ux+2vy+2wz+d=0$$

represents either a parabolic cylinder or a hyperbolic paraboloid.

9. If $\mathsf{F}(x, y, z)=0$ represents a cylinder, prove that

$$\frac{\dfrac{\partial \mathsf{S}}{\partial a}}{\dfrac{\partial \mathsf{D}}{\partial a}}=\frac{\dfrac{\partial \mathsf{S}}{\partial b}}{\dfrac{\partial \mathsf{D}}{\partial b}}=\frac{\dfrac{\partial \mathsf{S}}{\partial c}}{\dfrac{\partial \mathsf{D}}{\partial c}},$$

and that the area of a normal section is $\pi\dfrac{\dfrac{\partial \mathsf{S}}{\partial a}+\dfrac{\partial \mathsf{S}}{\partial b}+\dfrac{\partial \mathsf{S}}{\partial c}}{\left(\dfrac{\partial \mathsf{D}}{\partial a}+\dfrac{\partial \mathsf{D}}{\partial b}+\dfrac{\partial \mathsf{D}}{\partial c}\right)^{\frac{3}{2}}}$.

10. Prove that if $\mathsf{F}(x, y, z)=0$ represents a paraboloid of revolution, we have
$$agh+f(g^2+h^2)=bhf+g(h^2+f^2)=cfg+h(f^2+g^2)=0,$$
and that if it represents a right circular cylinder we have also
$$\frac{u}{f}+\frac{v}{g}+\frac{w}{h}=0.$$

11. The principal planes of $f(x, y, z)=1$ are given by
$$\begin{vmatrix} x, & y, & z \\ \dfrac{\partial f}{\partial x}, & \dfrac{\partial f}{\partial y}, & \dfrac{\partial f}{\partial z} \\ \dfrac{\partial \mathsf{F}}{\partial x}, & \dfrac{\partial \mathsf{F}}{\partial y}, & \dfrac{\partial \mathsf{F}}{\partial z} \end{vmatrix}=0,$$
where $\mathsf{F}(x, y, z)=0$ is the cone reciprocal to $f(x, y, z)=0$.

12. Prove that the centres of conicoids that touch $yz=mx$ at its vertex and at all points of its generator $y=kx$, $kz=m$, lie on the line $y=0$, $kz=m$.

13. Prove that $z(ax+by+cz)+\alpha x+\beta y=0$ represents a paraboloid and that the equations to the axis are
$$ax+by+2cz=0, \quad (a^2+b^2)z+a\alpha+b\beta=0.$$

14. A hyperbolic paraboloid passes through the lines $\dfrac{x}{-a}=\dfrac{y}{b}=\dfrac{z}{2c}$; $\dfrac{x}{a}=\dfrac{z}{c}-\dfrac{y}{b}=\dfrac{1}{2}$; and has one system of generators parallel to the plane $z=0$. Shew that the equations to the axis are
$$\frac{x}{a}-\frac{y}{b}+\frac{2z}{c}=0, \quad b^2\left(\frac{z}{c}-1\right)+a^2\left(\frac{z}{c}+1\right)=0.$$

15. Paraboloids are drawn through the lines $y=0$, $z=h$, $x=0$, $z=-h$; and touching the line $x=a$, $y=b$. Shew that their diameters through the point of contact lie on the conoid
$$a(y-b)(z-h)^2-b(x-a)(z+h)^2=0.$$

16. Given the ellipsoid of revolution
$$\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1, \quad (a^2>b^2).$$
Shew that the cone whose vertex is one of the foci of the ellipse $z=0$, $x^2/a^2+y^2/b^2=1$, and whose base is any plane section of the ellipsoid is of revolution.

17. The axes of the conicoids of revolution that pass through the six points $(\pm a, 0, 0)$, $(0, \pm b, 0)$, $(0, 0, \pm c)$ lie in the coordinate planes or on the cone
$$\frac{y^2-z^2}{a^2}+\frac{z^2-x^2}{b^2}+\frac{x^2-y^2}{c^2}=0.$$

18. Prove that the equation to the right circular cylinder on the circle through the three points $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$ is
$$(x^2+y^2+z^2-ax-by-cz)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1\right)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-2\right).$$

19 Find the equation to the paraboloid which has

$$y=z-k=0, \quad x=z+k=0$$

as generators and the other system parallel to the plane $x+y+z=0$. Find also the coordinates of the vertex and the equations to the axis.

20. The axes of cylinders that circumscribe an ellipsoid and have a cross-section of constant area lie on a cone concyclic with the ellipsoid.

21. A conicoid touches the plane $z=0$ and is cut by the planes $x=0$, $y=0$ in two circles of variable centres but constant radii a and b. Shew that the locus of the centre is

$$z^2(x^2-y^2)+a^2y^2-b^2x^2=0.$$

22. A, B, C are the points $(2a, 0, 0)$, $(0, 2b, 0)$, $(0, 0, 2c)$, and the axes are rectangular. A circle is circumscribed about the triangle OAB. A conicoid passes through this circle and is such that its sections by the planes $x=0$, $y=0$ are rectangular hyperbolas which pass through O, B and C; O, C and A respectively. Prove that the equation to the conicoid is

$$x^2+y^2-z^2+2\lambda yz+2\mu zx-2ax-2by+2cz=0,$$

where λ and μ are parameters, and that the locus of the centres of such conicoids is the sphere

$$x^2+y^2+z^2-ax-by-cz=0.$$

23. Shew that the equation to the conicoid that passes through the vertices of the tetrahedron whose faces are

$$x=0, \quad y=0, \quad z=0, \quad x/a+y/b+z/c=1,$$

and is such that the tangent plane at each vertex is parallel to the opposite face, is

$$\frac{x^2-ax}{a^2}+\frac{y^2-by}{b^2}+\frac{z^2-cz}{c^2}+\frac{1}{abc}(ayz+bzx+cxy)=0.$$

24. Shew that the equation to the ellipsoid inscribed in the tetrahedron whose faces are $x=0$, $y=0$, $z=0$, $x/a+y/b+z/c=1$, so as to touch each face at its centre of gravity, is

$$\frac{3x^2}{a^2}+\frac{3y^2}{b^2}+\frac{3z^2}{c^2}+\frac{3yz}{bc}+\frac{3zx}{ca}+\frac{3xy}{ab}-\frac{3x}{a}-\frac{3y}{b}-\frac{3z}{c}+1=0.$$

Shew that its centre is at the centre of gravity of the tetrahedron and that its equation referred to parallel axes through the centre is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}+\frac{yz}{bc}+\frac{zx}{ca}+\frac{xy}{ab}=\frac{1}{24}.$$

25. If the feet of the six normals from P to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

lie upon a concentric conicoid of revolution, the locus of P is the cone

$$\frac{y^2z^2}{a^2(b^2-c^2)}+\frac{z^2x^2}{b^2(c^2-a^2)}+\frac{x^2y^2}{c^2(a^2-b^2)}=0,$$

and the axes of symmetry of the conicoids lie on the cone

$$a^2(b^2-c^2)x^2+b^2(c^2-a^2)y^2+c^2(a^2-b^2)z^2=0.$$

26. If $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ represents a pair of planes, prove that the planes bisecting the angles between them are given by

$$\begin{vmatrix} x, & y, & z \\ ax+hy+gz, & hx+by+fz, & gx+fy+cz \\ F^{-1}, & G^{-1}, & H^{-1} \end{vmatrix}=0.$$

27. Prove that

$$\begin{aligned}(x^2+\alpha^2)(\beta+\gamma)+(y^2+\beta^2)(\gamma+\alpha)+(z^2+\gamma^2)(\alpha+\beta)\\ -2\alpha yz-2\beta zx-2\gamma xy+2x(2\beta\gamma-\alpha\beta-\alpha\gamma)\\ +2y(2\gamma\alpha-\beta\gamma-\beta\alpha)+2z(2\alpha\beta-\gamma\alpha-\gamma\beta)=0\end{aligned}$$

represents a cylinder whose axis is

$$x-\alpha=y-\beta=z-\gamma.$$

CHAPTER XII.

THE INTERSECTION OF TWO CONICOIDS.

164. Any plane meets a conicoid in a conic, and therefore any plane meets the curve of intersection of two conicoids in the four points common to two conics. The curve of intersection is therefore of the fourth degree, or is a **quartic curve.**

If the conicoids have a common generator, any plane which does not pass through it meets it in one point and meets the locus of the other common points of the conicoids in three points, and therefore the locus is a **cubic curve.** Thus the quartic curve of intersection of two conicoids may consist of a straight line and a cubic curve.

Ex. The conicoids $zx=y^2$, $xy=z$ have **OX** as a common generator. Their other common points lie on a cubic curve whose equations may be written $x=t$, $y=t^2$, $z=t^3$, where t is a parameter.

Again, an asymptote of one of the two conics, in which a given plane cuts two conicoids, may be parallel to an asymptote of the other. In that case the conics will intersect in three points at a finite distance, and the locus of the common points of the two conicoids which are at a finite distance will be a cubic curve.

Ex. We have seen that three cylinders pass through the feet of the normals from a point (α, β, γ) to the conicoid

$$ax^2+by^2+cz^2=1.$$

Their equations are $$\frac{x-\alpha}{ax}=\frac{y-\beta}{by}=\frac{z-\gamma}{cz},$$

or $$yz(b-c)-b\gamma y+c\beta z=0,\quad zx(c-a)-c\alpha z+a\gamma x=0,$$
$$xy(a-b)-a\beta x+b\alpha y=0.$$

Their curve of intersection is a cubic curve whose equations may be written

$$x=\frac{\alpha}{1-at},\quad y=\frac{\beta}{1-bt},\quad z=\frac{\gamma}{1-ct}.$$

One asymptote of any plane section of the first lies in the plane

$$z-\frac{b\gamma}{b-c}=0,$$

and one asymptote of any plane section of the second lies in the plane

$$z+\frac{a\gamma}{c-a}=0.$$

Hence any plane meets the two cylinders in two conics such that an asymptote of one is parallel to an asymptote of the other, and the conics therefore intersect in three points at a finite distance.

165. The cubic curve common to two conicoids. Suppose that the locus of the common points of two conicoids S_1 and S_2 consists of a common generator AB and a cubic curve. Any generator, PQ, of S_1, of the opposite system to AB, meets S_2 in two points, one of which lies upon AB and the

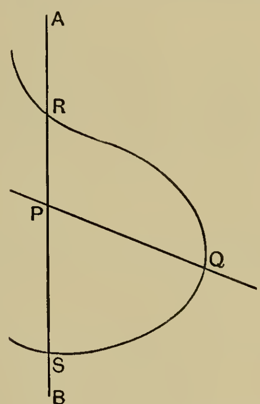

FIG. 49.

other upon the curve. Let P, fig. 49, be the first of these points and Q the second. The plane containing AB and PQ meets the curve in three points, one of which is Q. But all points of the curve lie upon S_1 and the plane intersects S_1 in the lines AB and PQ, therefore the other two points must lie upon AB or PQ, or one upon AB and one upon PQ. Neither can lie upon PQ, for then PQ would meet the surface S_2 in three points, and would therefore be a generator

of S_2. Therefore the cubic curve intersects the common generator AB at two points.

Let AB meet the curve in R and S, and let P move along AB. As P tends to R, Q tends to P, so that in the limit PQ is a tangent at R to the surface S_2, and the plane of AB and PQ is then the tangent plane at R to the surface S_2. But the plane of AB and PQ is tangent plane at P to the surface S_1 for any position of P. And therefore the surfaces S_1 and S_2 have the same tangent plane at R. Similarly, the surfaces also touch at S. But we have proved, (§ 134, Ex. 10), that if two conicoids have a common generator, they touch at two points of the generator. Hence the locus of the common points of two conicoids which have a common generator consists of the generator and a cubic curve, which passes through the two points of the generator at which the surfaces touch.

Ex. 1. The conicoids $5x^2-yz-2zx+2xy+2x+2y=0, \quad \text{............(1)}$

$$2x^2-zx+x+y=0, \quad \text{.............................(2)}$$

have OZ as a common generator. Any plane through the generator is given by $y=tx$. To find where this plane meets the cubic curve common to the conicoids, substitute in equations (1) and (2). We obtain

$$x=0, \quad x(5+2t)-z(t+2)+2(t+1)=0, \quad \text{..................(3)}$$

$$x=0, \quad 2x \quad -z \quad + \quad t+1 \;=0. \quad \text{..................(4)}$$

The points corresponding to $x=0$ lie upon the common generator. The remaining point of intersection of the plane and cubic has, from (3) and (4), coordinates

$$x=t(t+1), \quad z=(2t+1)(t+1)\,; \quad \text{and} \quad y=tx=t^2(t+1). \quad \text{........(5)}$$

But t is a variable parameter, so that we may take the equations (5) to represent the curve. The points where the curve meets the common generator OZ are given by $t=0$, $t=-1$. They are the points (0, 0, 1), (0, 0, 0). It is easy to verify that the common tangent planes at these points are $y=0$, $x+y=0$.

Ex. 2. Prove that the conicoids

$$x^2-y^2-yz+zx+x-2y+z=0,$$

$$x^2+2y^2+3z^2-3yz+zx-4xy+x-2y+z=0$$

have $x=y=z$ as a common generator. Prove that the plane

$$x-y=t(y-z)$$

meets the cubic curve which contains the other common points in the point

$$x=\frac{(4t^2+4t+3)(1-t)}{4t^3+5t}, \quad y=\frac{3(1-t)}{4t^3+5t}, \quad z=\frac{(4t+1)(t-1)}{4t^3+5t}\,;$$

shew that the cubic meets the common generator at the origin and the point $x=y=z=-\frac{2}{3}$, and verify that the surfaces have the same tangent planes at these points.

166. Conicoids with common generators. The cubic curve may degenerate into a straight line and a conic or into three straight lines.

Let O and P, (fig. 50), be the points of the common generator at which the surfaces touch and let the measure of OP be γ. Take OP as z-axis and O as origin. Let OX and PG be the other generators of the conicoid S_1 which pass through O and P. Take OX as x-axis, and the parallel through O to PG as y-axis. Then, since

$$x=0,\ y=0; \quad y=0,\ z=0; \quad z=\gamma,\ x=0$$

are generators of S_1, its equation may be written

$$2yz+2gzx+2hxy-2\gamma y=0. \qquad \text{...............(1)}$$

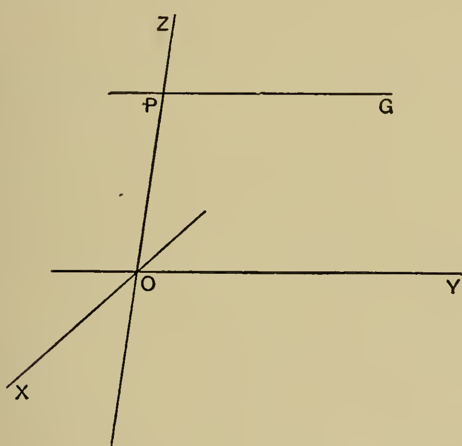

FIG. 50.

And since the tangent planes at the origin and $(0, 0, \gamma)$ to S_2 are $y=0$, $x=0$ respectively, the equation to S_2 is

$$a_1x^2+b_1y^2+2yz+2g_1zx+2h_1xy-2\gamma y=0. \qquad \text{......(2)}$$

The tangent planes at $(0, 0, z')$ to S_1 and S_2 are given by

$$gz'x-y(\gamma-z')=0, \quad g_1z'x-y(\gamma-z')=0,$$

and hence if $g=g_1$ the surfaces touch at all points of the

common generator OZ. We shall consider meantime the case where $g \neq g_1$. From (1) and (2), by subtracting, we obtain

$$a_1x^2+b_1y^2+2zx(g_1-g)+2xy(h_1-h)=0, \quad \ldots\ldots\ldots(3)$$

which clearly represents a surface through the common points of S_1 and S_2. It is in general a cone, having OZ as a generator, and in general, the locus of the common points of S_1 and S_2 is a cubic curve which lies upon the cone. But if equation (3) represents two intersecting planes, the cubic will degenerate. The condition for a pair of planes is

$$b_1(g_1-g)^2=0, \quad \text{and hence} \quad b_1=0.$$

If $b_1=0$, PG, whose equations are $x=0$, $z=\gamma$, is a generator of S_2, and equation (3) then becomes equivalent to

$$x=0, \quad a_1x+2z(g_1-g)+2y(h_1-h)=0.$$

Hence the common points of S_1 and S_2 lie upon a conic in the plane

$$a_1x+2z(g_1-g)+2y(h_1-h)=0,$$

and the two common generators, OZ and PG, in the plane $x=0$.

If, also, $a_1=0$, OX is generator of S_2. The plane of the conic then passes through OX, and is therefore a tangent plane to both conicoids. The conic therefore becomes two straight lines, one of which is OX, and the other a generator of the opposite system. But OX and PG are of the same system, and the conic consists therefore of OX and a generator which intersects OX and PG. The complete locus of the common points of S_1 and S_2 is then a skew quadrilateral formed by four common generators.

If the conicoids touch at all points of the common generator OZ, $g=g_1$, and equation (3) becomes

$$a_1x^2+2(h_1-h)xy+b_1y^2=0,$$

which represents a pair of planes through OZ.

If these planes are distinct, they meet the conicoids in two other common generators of the opposite system to OZ. If they are coincident the conicoids touch at all points of a second common generator.

Ex. 1. The conicoids

$$2x^2-y^2+4z^2+3yz+6zx+4xy-2x+y-4z=0,$$
$$2x^2-y^2-5z^2-6yz-3zx+4xy-2x+y+5z=0$$

have two common generators and a common conic section.

(The generators are $x=0$, $y+z=1$; $y=0$, $z+x=1$.)

Ex. 2. The conicoids

$$3y^2+4z^2+6yz-5zx-xy+y+z=0,$$
$$4y^2+6z^2+9yz-3zx+2y+3z=0$$

have **OX** for a common generator. Find the locus of their other common points.

Ans. $3x+1=-2y=4z$, and $x+y+z+1=0$, $(2y+3z)^2+2y+6z=0$.

Ex. 3. The conicoids

$$2z^2-3yz-5zx-6xy+z=0,$$
$$4z^2-6yz-10zx+7xy+2z=0$$

have four common generators.

($y=0$, $z=0$; $z=0$, $x=0$; $x=0$, $3y-2z=1$; $y=0$, $5x-2z=1$.)

Ex. 4. The conicoids

$$z^2+2yz+6zx-3xy-12z=0,$$
$$4z^2-2yz-4zx+2xy+8z=0$$

have two common generators and touch at all points of these generators.

Ex. 5. Prove that the intersection of the conicoids

$$z^2+2z-y+2=0, \quad y^2-2y-x-1=0$$

is a quartic curve whose equations may be written

$$x=\lambda^4-2, \quad y=\lambda^2+1, \quad z=\lambda-1.$$

Ex. 6. Find the points of intersection of the plane $x-9y-4z=0$ and the quartic curve which is common to the parabolic cylinders

$$z^2+10z-y+26=0, \quad y^2-2y-x+2=0.$$

Ans. Two coincident at $(17, 5, -7)$; $(2, 2, -4)$; $(82, 10, -2)$.

Ex. 7. Prove that the conicoids

$$3x^2+4z^2-4yz-zx-2xy-2x+2z=0,$$
$$x^2-y^2-8z^2+7yz+12zx-11xy-2x+2z=0$$

touch one another at all points of the common generator $x=y=z$, and that their other common generators lie in the planes

$$2(x-y)^2+13(x-y)(y-z)+12(y-z)^2=0.$$

Ex. 8. If two cones have a common generator, their other common points lie on a cubic curve which passes through both vertices.

Ex. 9. If two paraboloids have each a system of generators parallel to a given plane and touch at two points of a common generator of the system, they touch at all points of the generator.

Ex. 10. Prove that the three cylinders

$$y(a-z)=a^2, \quad z(a-x)=a^2, \quad x(a-y)=a^2$$

pass through a cubic curve which lies on the surface $xyz+a^3=0$.

Ex. 11. Prove that if the cubic curve

$$x=\frac{2}{t-a}, \quad y=\frac{2}{t-b}, \quad z=\frac{2}{t-c}$$

meets a conicoid in seven points, it lies wholly on the conicoid.

Shew that the curve lies upon the cylinders

$$\mathbf{C}_1 \equiv yz(b-c)-2y+2z=0,$$
$$\mathbf{C}_2 \equiv zx(c-a)-2z+2x=0,$$
$$\mathbf{C}_3 \equiv xy(a-b)-2x+2y=0,$$

and hence that the general equation to a conicoid through it is

$$\lambda\mathbf{C}_1+\mu\mathbf{C}_2+\nu\mathbf{C}_3=0.$$

Prove that the locus of the centres of conicoids that pass through the curve is

$$(b-c)(c-a)(a-b)xyz+\Sigma(b-c)(b+c-2a)yz-2\Sigma(b-c)x=0,$$

and that this surface is also the locus of the mid-points of chords of the curve.

Shew that the lines

$$y=\frac{2}{a-b}, \; z=\frac{2}{a-c}; \quad z=\frac{2}{b-c}, \; x=\frac{2}{b-a}; \quad x=\frac{2}{c-a}, \; y=\frac{2}{c-b}$$

are asymptotic to the curve, and that the locus of the centres passes through them and through the curve.

Ex. 12. Prove that the general equation to a conicoid through the cubic curve given by

$$x=t, \quad y=t^2, \quad z=t^3$$

is

$$\lambda(xy-z)+\mu(zx-y^2)+\nu(x^2-y)=0,$$

and that the locus of the centres of such conicoids is the surface

$$2x^3-3xy+z=0.$$

Verify that this surface is also the locus of the mid-points of chords of the curve.

Ex. 13. The equations to a cubic curve are

$$x=a_1t^3+b_1t^2+c_1t, \quad y=a_2t^3+b_2t^2+c_2t, \quad z=a_3t^3+b_3t^2+c_3t.$$

Prove that the cone generated by chords through the origin is given by $wu=v^2$, where

$$u\equiv \mathbf{A}_1x+\mathbf{A}_2y+\mathbf{A}_3z, \quad v\equiv \mathbf{B}_1x+\mathbf{B}_2y+\mathbf{B}_3z, \quad w\equiv \mathbf{C}_1x+\mathbf{C}_2y+\mathbf{C}_3z;$$

$$\mathbf{A}_1=\frac{\partial\Delta}{\partial a_1}, \quad \text{etc.}; \quad \Delta\equiv\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}.$$

Shew that the curve lies on each of the conicoids, (two of which have a common generator),

$$wu=v^2, \quad vw=\Delta u, \quad w^2=\Delta v,$$

and that the locus of the centres of conicoids that pass through it is

$$2w(w^2-\Delta v)-\Delta(vw-\Delta u)=0.$$

Ex. 14. Prove that the equations

$$x=\frac{a_1t+b_1}{\alpha_1t+\beta_1},\quad y=\frac{a_2t+b_2}{\alpha_2t+\beta_2},\quad z=\frac{a_3t+b_3}{\alpha_3t+\beta_3}$$

determine a cubic curve, which lies upon three cylinders whose generators are parallel to the coordinate axes.

Ex. 15. Prove that the cubic curve given by

$$x=\frac{a_1t^3+b_1t^2+c_1t+d_1}{a_4t^3+b_4t^2+c_4t+d_4},\quad y=\frac{a_2t^3+b_2t^2+c_2t+d_2}{a_4t^3+b_4t^2+c_4t+d_4},\quad z=\frac{a_3t^3+b_3t^2+c_3t+d_3}{a_4t^3+b_4t^2+c_4t+d_4},$$

lies upon the conicoids

$$u_3u_1=u_2^2,\quad u_4u_2=u_3^2,\quad u_2u_3=u_1u_4,$$

where $\quad u_1\equiv \mathsf{A}_1x+\mathsf{A}_2y+\mathsf{A}_3z+\mathsf{A}_4,\quad u_2\equiv \mathsf{B}_1x+\mathsf{B}_2y+\mathsf{B}_3z+\mathsf{B}_4,$

$$u_3\equiv \mathsf{C}_1x+\mathsf{C}_2y+\mathsf{C}_3z+\mathsf{C}_4,\quad u_4\equiv \mathsf{D}_1x+\mathsf{D}_2y+\mathsf{D}_3z+\mathsf{D}_4\,;$$

$$\mathsf{A}_1=\frac{\partial\Delta}{\partial a_1},\ \text{etc.};\quad \Delta\equiv\begin{vmatrix} a_1, & b_1, & c_1, & d_1\\ a_2, & b_2, & c_2, & d_2\\ a_3, & b_3, & c_3, & d_3\\ a_4, & b_4, & c_4, & d_4\end{vmatrix}.$$

If A, B, C, D are the points

$$\left(\frac{a_1}{a_4},\ \frac{a_2}{a_4},\ \frac{a_3}{a_4}\right),\quad \left(\frac{b_1}{b_4},\ \frac{b_2}{b_4},\ \frac{b_3}{b_4}\right),\quad \left(\frac{c_1}{c_4},\ \frac{c_2}{c_4},\ \frac{c_3}{c_4}\right),\quad \left(\frac{d_1}{d_4},\ \frac{d_2}{d_4},\ \frac{d_3}{d_4}\right),$$

each of the conicoids passes through two of the lines BA, AD, DC.

The equations $u_3u_1=u_2^2$, $u_4u_2=u_3^2$ represent cones whose vertices are D and A respectively.

The curve passes through A and D and touches BA at A and DC at D.

The centres of conicoids through it lie on the cubic surface

$$2(a_4u_2+c_4u_4)(u_3u_1-u_2^2)-2(b_4u_1+d_4u_3)(u_4u_2-u_3^2)$$
$$+(a_4u_1-b_4u_2+c_4u_3-d_4u_4)(u_2u_3-u_1u_4)=0.$$

167. The cones through the intersection of two conicoids. *Four cones, in general, pass through the curve of intersection of two given conicoids, and their vertices are the summits of a tetrahedron which is self-polar with respect to any conicoid through the curve of intersection.*

If the equations to the conicoids are

$$\mathsf{S}\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy$$
$$+2ux+2vy+2wz+d=0,$$

$$\mathsf{S}'\equiv a'x^2+b'y^2+c'z^2+2f'yz+2g'zx+2h'xy$$
$$+2u'x+2v'y+2w'z+d'=0,$$

the equation $S+\lambda S'=0$ represents a conicoid through the curve of intersection. This conicoid is a cone if

$$\begin{vmatrix} a+\lambda a', & h+\lambda h', & g+\lambda g', & u+\lambda u' \\ h+\lambda h', & b+\lambda b', & f+\lambda f', & v+\lambda v' \\ g+\lambda g', & f+\lambda f', & c+\lambda c', & w+\lambda w' \\ u+\lambda u', & v+\lambda v', & w+\lambda w', & d+\lambda d' \end{vmatrix}=0,$$

and this equation gives four values of λ. If these are $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, then (α, β, γ), the vertex of the cone corresponding to λ_1, is given by

$$S_\alpha+\lambda_1 S'_\alpha=0,\ S_\beta+\lambda_1 S'_\beta=0,\ S_\gamma+\lambda_1 S'_\gamma=0,\ S_t+\lambda_1 S'_t=0, \quad (1)$$

where $S_\alpha \equiv \dfrac{\partial S}{\partial \alpha}$, etc. Again, the polar plane of (α, β, γ) with respect to $S+\mu S'=0$ has for equation

$$x(S_\alpha+\mu S'_\alpha)+y(S_\beta+\mu S'_\beta)+z(S_\gamma+\mu S'_\gamma)+(S_t+\mu S'_t)=0,$$

which by means of the relations (1) reduces to

$$(\mu-\lambda_1)(xS'_\alpha+yS'_\beta+zS'_\gamma+S'_t)=0.$$

The polar plane of (α, β, γ) with respect to any conicoid through the curve of intersection is therefore the polar plane with respect to the conicoid S'. Hence this plane is the polar plane of (α, β, γ) with respect to the three cones corresponding to $\lambda_2, \lambda_3, \lambda_4$, and therefore passes through the vertices of these cones. Thus the plane through the vertices of any three of the cones is the polar plane of the fourth vertex with respect to any conicoid of the system, or the four vertices form a self-polar tetrahedron.

168. Conicoids with double contact. If two conicoids have common tangent planes at two points they are said to have **double contact.**

If two conicoids have double contact and the line joining the points of contact is not a common generator, their curve of intersection consists of two conics.

If the points of contact are A and B, any plane through AB meets the conicoids in two conics which touch at A and B. Take AB as y-axis and any two lines through

a point on AB as x- and z-axes. Let the conics in which the xy-plane cuts the conicoids be

$$f \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

and
$$f + \lambda x^2 = 0.$$

Then the equations to the two conicoids are

$$f + z(lx + my + nz + p) = 0, \quad \text{............(1)}$$

$$f + \lambda x^2 + z(l'x + m'y + n'z + p') = 0. \quad \text{............(2)}$$

But the sections of the conicoids by the yz-plane also touch at A and B, and therefore their equations are of the forms
$$\phi(y, z) = 0, \quad \phi(y, z) + \lambda' z^2 = 0.$$

The sections of the conicoids by the plane $x = 0$ are given by
$$by^2 + 2fy + c + z(my + nz + p) = 0,$$
$$by^2 + 2fy + c + z(m'y + n'z + p') = 0,$$

and therefore
$$m = m' \text{ and } p = p'.$$

From (1) and (2), by subtraction, we have

$$\lambda x^2 + z\{(l' - l)x + (n' - n)z\} = 0:$$

therefore the common points of the two conicoids lie in two planes which pass through AB, or the curve of intersection consists of two conics which cross at A and B.

If AB is a common generator of the two conicoids, the other common points lie on a cubic curve, which may, as we have seen in § 166, consist of a straight line and a conic, or three straight lines. In either case the common points lie in two planes. In the first case, if the common generators AB and AC meet the conic in B and C, the conicoids touch at the three points A, B, C. For the tangent plane to either conicoid at B is the plane containing AB and the tangent to the conic at B, and the tangent plane to either conicoid at C is the plane containing AC and the tangent to the conic at C; also the plane BAC is the common tangent plane at A. In the second case, the common points of the conicoid lie on the sides of a skew quadrilateral and the conicoids touch at the four vertices.

169. *If two conicoids have two common plane sections they touch at two points, at least.*

The line of intersection of the planes of the sections will meet the conicoids in two common points A and B. The tangents to the sections at A are tangents to both conicoids at A, and therefore, since two tangents determine the tangent plane at any point, the conicoids have the same tangent plane at A. Similarly they touch at B. If one plane section consists of two generators CA and CB, the conicoids also touch at C. If the other also consists of two generators the conicoids touch at their point of intersection, and thus touch at four points.

The analytical proof is equally simple. If $S=0$ is the equation to one conicoid, and

$$u \equiv ax+by+cz+d=0, \quad v \equiv a'x+b'y+c'z+d'=0$$

represent the planes of the common sections, the equation to the other conicoid is of the form

$$S+\lambda uv=0.$$

If A is the point (α, β, γ), then

$$u' \equiv a\alpha+b\beta+c\gamma+d=0, \text{ and } v' \equiv a'\alpha+b'\beta+c'\gamma+d'=0. \quad (1)$$

The equation to the tangent plane at A to the second conicoid is

$$xS_\alpha+yS_\beta+zS_\gamma+S_t+\lambda(uv'+vu')=0,$$

or, by (1), $$xS_\alpha+yS_\beta+zS_\gamma+S_t=0,$$

which represents the tangent plane at A to the first conicoid. Hence the conicoids touch at A, and similarly they touch at B.

170. The general equation to a conicoid having double contact with $S=0$, the chord of contact being $u=0$, $v=0$, is

$$S+\lambda u^2+2\mu uv+\nu v^2=0.$$

For the tangent plane at A is

$$xS_\alpha+yS_\beta+zS_\gamma+S_t+2\lambda uu'+2\mu(uv'+vu')+2\nu vv'=0,$$

or, since $\quad u'=v'=0, \quad xS_\alpha+yS_\beta+zS_\gamma+S_t=0.$

Thus the conicoids touch at A, and similarly, at B. Again, three conditions must be satisfied if a conicoid is to

touch a given plane at a given point, and therefore the general equation should contain three disposable constants, which it does.

Cor. A focus of a conicoid is a sphere of zero radius which has double contact with the conicoid, and the corresponding directrix is the chord of contact.

171. Circumscribing conicoids. If two conicoids touch at three points A, B, C and none of the lines BC, CA, AB is a common generator, then the conicoids touch at all points of their sections by the plane ABC.

Since the conicoids touch at B and C, their common points lie in two planes which pass through BC, (§ 168). Since these planes pass through A, they must coincide in the plane ABC. The curve of intersection of the surfaces consists therefore of two coincident conics in the plane ABC, and the surfaces touch at points of their section by the plane.

When two conicoids touch at all points of a plane section one is said to be **circumscribed** to the other.

Ex. 1. If two conicoids have a common plane section, their other points of intersection lie in one plane.

Ex. 2. If three conicoids have a common plane section, the planes of their other common sections pass through one line.

Ex. 3. The locus of a point such that the square on the tangent from it to a given sphere is proportional to the rectangle contained by its distances from two given planes is a conicoid which has double contact with the sphere.

Ex. 4. Two conicoids which are circumscribed to a third intersect in plane curves.

Ex. 5. When three conicoids are circumscribed to a fourth, they intersect in plane curves, and certain sets of three of the six planes of intersection, one from each pair of conicoids, pass through one line.

Ex. 6. Prove that the ellipsoid and sphere given by

$$x^2+5y^2+14z^2=200, \quad 5(x^2+y^2+z^2)-64x+36z+20=0$$

have double contact, and that the chord of contact is $x=8$, $z=2$.

Ex. 7. If a system of conicoids has a common conic section, the polar planes with respect to them of any point in the plane of the section pass through one line.

Ex. 8. If two conicoids have two common generators of the same system, they have two other common generators.

Ex. 9. The centres of conicoids which have double contact with the surface

$$ax^2+by^2+cz^2=1$$

at its points of intersection with the chord $x=\alpha$, $y=\beta$, and intersect the plane $z=0$ in a circle, lie on the line

$$z=0, \quad \frac{x}{a\alpha}-\frac{y}{b\beta}=\frac{1}{a}-\frac{1}{b}.$$

Ex. 10. A sphere of constant radius r has double contact with the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Prove that its centre must lie on one of the conics

$$x=0, \quad \frac{y^2}{b^2-a^2}+\frac{z^2}{c^2-a^2}=1-\frac{r^2}{a^2};$$

$$y=0, \quad \frac{z^2}{c^2-b^2}+\frac{x^2}{a^2-b^2}=1-\frac{r^2}{b^2};$$

$$z=0, \quad \frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}=1-\frac{r^2}{c^2}.$$

Examine when the contact is real and when the sphere lies wholly within the ellipsoid. Cf. § 130.

Ex. 11. If a conicoid is circumscribed to a sphere, every tangent plane to the sphere cuts the conicoid in a conic which has a focus at the point of contact.

Ex. 12. If a conicoid is circumscribed to another conicoid, the tangent plane to either at an umbilic cuts the other in a conic of which the umbilic is a focus.

Ex. 13. Any two enveloping cones of the conicoid

$$ax^2+by^2+cz^2=1$$

whose vertices lie on the concentric and homothetic conicoid

$$ax^2+by^2+cz^2=k^2$$

have double contact.

Ex. 14. The centres of conicoids which have double contact with a given conicoid so that the chord of contact is parallel to a given line lie in a given plane.

Ex. 15. If two cones have a common circular section, they have double contact, and if the line joining their vertices meet the plane of the circle in P, the chord of contact is the polar of P with respect to the circle.

Ex. 16. If a sphere has double contact with an ellipsoid, the chord of contact is parallel to one of the principal axes, and the angle between the planes of the common sections of the sphere and the ellipsoid is the same for all chords parallel to a given axis.

172. Conicoids through eight given points. An infinite number of conicoids can be found to pass through eight given points.

Take A and B any other two fixed points. Then one conicoid can be found to pass through A and the eight given points, and one to pass through B and the eight given points. Let the equations to these conicoids be $S=0$, $S'=0$. The equation $S+\lambda S'=0$ represents a conicoid which passes through all the points common to the conicoids given by $S=0$, $S'=0$, and therefore through the eight given points. And any value can be assigned to the parameter λ; therefore an infinite number of conicoids can be found to pass through the eight given points.

The locus of the common points of $S=0$, $S'=0$ is a quartic curve. Hence all conicoids through eight given points pass through a quartic curve.

Cor. One conicoid, in general, passes through nine given points, but if the ninth point lies on the quartic curve through the other eight, an infinite number of conicoids passes through the nine.

173. *The polar planes of a given point with respect to the conicoids through eight given points pass through a fixed line.*

Any conicoid through the eight points is given by $S+\lambda S'=0$, where $S=0$ and $S'=0$ represent fixed conicoids through the points. The equation to the polar plane of (α, β, γ) with respect to the conicoid $S+\lambda S'=0$ is

$$xS_\alpha+yS_\beta+zS_\gamma+S_t+\lambda(xS'_\alpha+yS'_\beta+zS'_\gamma+S'_t)=0.$$

Hence, whatever the value of λ, the polar plane passes through the fixed line

$$xS_\alpha+yS_\beta+zS_\gamma+S_t=0, \quad xS'_\alpha+yS'_\beta+zS'_\gamma+S'_t=0.$$

Ex. 1. If four conicoids pass through eight given points, the polar planes of any point with respect to them have the same anharmonic ratio.

Ex. 2. The diametral planes of a given line with respect to the conicoids through eight given points pass through a fixed line.

Ex. 3. The polars of a given line with respect to the conicoids through eight given points lie on a hyperboloid of one sheet.

If A_1, $(\alpha_1, \beta_1, \gamma_1)$ and A_2, $(\alpha_2\ \beta_2, \gamma_2)$ are points of the given line, and we denote
$$xS_{\alpha_1}+yS_{\beta_1}+zS_{\gamma_1}+S_{t_1} \text{ by } P_{\alpha_1}$$
and
$$xS_{\alpha_2}+yS_{\beta_2}+zS_{\gamma_2}+S_{t_2} \text{ by } P_{\alpha_2},$$
then the equations to the polar of A_1A_2 with respect to the conicoid $S+\lambda S'=0$ are
$$P_{\alpha_1}+\lambda P'_{\alpha_1}=0, \quad P_{\alpha_2}+\lambda P'_{\alpha_2}=0.$$

The locus of the polars is therefore given by
$$P_{\alpha_1}P'_{\alpha_2}-P_{\alpha_2}P'_{\alpha_1}=0.$$

Ex. 4. The pole of a given plane with respect to the conicoids through eight given points lies on a cubic curve, the intersection of two hyperboloids which have a common generator.

Let A_1, $(\alpha_1, \beta_1, \gamma_1)$, A_2, $(\alpha_2, \beta_2, \gamma_2)$, A_3, $(\alpha_3, \beta_3, \gamma_3)$ be three points of the fixed plane. Then the po e of the fixed plane with respect to the conicoid $S+\lambda S'=0$ is the point of intersection of the polar planes of A_1, A_2, A_3, and therefore is given by
$$P_{\alpha_1}+\lambda P'_{\alpha_1}=0, \quad P_{\alpha_2}+\lambda P'_{\alpha_2}=0, \quad P_{\alpha_3}+\lambda P'_{\alpha_3}=0.$$

The locus of the pole is therefore the curve of intersection of the hyperboloids
$$P_{\alpha_2}P'_{\alpha_3}-P_{\alpha_3}P'_{\alpha_2}=0, \quad P_{\alpha_3}P'_{\alpha_1}-P_{\alpha_1}P'_{\alpha_3}=0.$$

Ex. 5. The centres of conicoids that pass through eight given points lie on a cubic curve.

174. Conicoids through seven given points. If $S=0$, $S'=0$, $S''=0$, are the equations to fixed conicoids through the seven given points, the general equation to a conicoid through the points is
$$S+\lambda S'+\mu S''=0. \quad \text{.......................}(1)$$

The fixed conicoids intersect in eight points whose coordinates are given by $S=0$, $S'=0$, $S''=0$, and therefore evidently satisfy the equation (1). Therefore all conicoids which pass through seven given points pass through an eighth fixed point.

Ex. 1. The polar planes of a given point with respect to the conicoids which pass through seven given points pass through a fixed point.

Ex. 2. The diametral planes of a fixed line with respect to the coincoids which pass through seven given points pass through a fixed point.

Ex. 3. The poles of a given plane with respect to the conicoids which pass through seven given points lie on a surface of the third degree.

Ex. 4. The centres of the conicoids which pass through seven given points lie on a surface of the third degree.

Ex. 5. The vertices of the cones that pass through seven given points lie on a curve of the sixth degree.

Examples IX.

1. Tangent planes parallel to a given plane are drawn to a system of conicoids which have double contact at fixed points with a given conicoid. Prove that the locus of their points of contact is a hyperbolic paraboloid which has one system of generators parallel to the given plane.

2. Tangent planes are drawn through a given line to a system of conicoids which have contact with a given conicoid at fixed points A and B. Prove that the locus of the points of contact is a hyperboloid which passes through A and B.

3. The feet of the normals to a conicoid from points on a given straight line lie on a quartic curve.

4. The edges OA, OB, OC of a parallelepiped are fixed in position, and the diagonal plane ABC passes through a fixed line. Prove that the vertex opposite to O lies on a cubic curve which lies on a cone that has OA, OB, OC as generators.

5. A variable plane ABC passes through a fixed line and cuts the axes, which are rectangular, in A, B, C. Prove that the locus of the centre of the sphere OABC is a cubic curve.

6. The feet of the perpendiculars from a point (α, β, γ) to the generators of the paraboloid $xy=cz$ lie on two cubic curves whose equations may be written

$$x=\frac{\gamma t+\alpha t^2}{1+t^2}, \quad y=\frac{c}{t}, \qquad z=\frac{\gamma+\alpha t}{1+t^2};$$

$$x=\frac{c}{t}, \qquad y=\frac{\gamma t+\beta t^2}{1+t^2}, \quad z=\frac{\gamma+\beta t}{1+t^2}.$$

7. The shortest distance between the fixed line $x=a$, $z=b$, and the generator $y=\lambda$, $\lambda x=z$, of the paraboloid $xy=z$, meets the generator in P. Shew that the locus of P is a cubic curve which lies on the cylinder

$$x^2+z^2-ax-bz=0.$$

8. Find the locus of the centres of conicoids that pass through a given conic and a straight line which intersects the conic.

9. Two cones have their vertices at an umbilic of an ellipsoid and meet the tangent plane at the opposite umbilic in two circles which cut at right angles. Shew that their curves of intersection with the ellipsoid lie in two planes, each of which contains the pole of the other.

10. If a cone with a given vertex P has double contact with a given conicoid, the chord of contact lies in the polar plane of P with respect to the conicoid.

11. A variable plane ABC meets the axes in A, B, C, and is at a constant distance p from the origin. A cone passes through the curves of intersection of the ellipsoid whose semiaxes are OA, OB, OC, and the planes OBC, ABC. Prove that its vertex lies on the surface

$$(1 \pm \sqrt{2})^2 x^{-2} + y^{-2} + z^{-2} = p^{-2}.$$

12. When two conicoids touch at all points of a common generator AB, the line joining the poles of any given plane with respect to them intersects AB.

13. AB is a given chord of a cubic curve. Prove that an infinite number of conicoids can be found to pass through the curve and through AB, and that one of these will touch a given plane which passes through AB at a given point of AB.

The z-axis is a chord of the curve

$$x = t^2 + t, \quad y = t^3 + t^2, \quad z = 2t^2 + 3t + 1.$$

Prove that the equation to the conicoid which passes through the curve and the z-axis and which touches the plane $2x = 3y$ at the point (0, 0, 2) is

$$7x^2 + yz - 4zx - 2xy + 4x + 4y = 0.$$

14. Prove that the conicoids

$$2x^2 - y^2 - z^2 + 2yz - 2xy + 2x - 2y = 0,$$

$$x^2 - y^2 - yz + 3zx - 2xy - 2y + 2z = 0$$

have a common generator $x = y = z$, and pass through the cubic curve

$$x = \frac{-2(2t^3 + t^2 + 1)}{4t^3 + 2t^2 - 3t + 2}, \quad y = \frac{2(t^3 - t^2 - 1)}{4t^3 + 2t^2 - 3t + 2}, \quad z = \frac{2(t^3 + 2t^2 - 1)}{4t^3 + 2t^2 - 3t + 2},$$

which touches the generator at $(-1, -1, -1)$.

15. If two conicoids, C_1 and C_2, have double contact, and the pole with respect to C_1 of one of the planes of the common sections lies on C_2, then the pole of the other also lies on C_2.

16. Find the locus of the centres of conicoids of revolution that circumscribe a given ellipsoid and pass through its centre.

17. P is any point on the curve of intersection of two right cones whose axes are parallel and whose semivertical angles are α and α'. If d and d' are the distances of P from the vertices, prove that $d \cos \alpha \pm d' \cos \alpha'$ is constant.

18. If a variable conicoid has double contact with each of three confocals it has a fixed director sphere.

19. Prove that two paraboloids can be drawn to pass through a given small circle on a given sphere and to touch the sphere at a given point, and prove that their axes are coplanar.

20. OP and OQ are the generators of a hyperboloid through a point O on the director sphere. Prove that the two paraboloids which contain the normals to the hyperboloid at points on OP and OQ intersect in a cubic curve whose projection on the tangent plane at O is a plane cubic with three real asymptotes.

21. The sides of a skew quadrilateral are the x-axis, the y-axis, and the lines

$$y=0, \quad lx+mz+1=0; \quad x=0, \quad l'y+mz+1=0.$$

Prove that the general equation to a conicoid which touches the sides is

$$z(lx+\alpha y+mz+1)+\kappa xy+(\lambda x+\mu y+\nu z+\rho)^2=0,$$

where $\alpha=l'$ or $l'-4(\mu-l'\rho)(\nu-m\rho)$.

22. Give a geometrical interpretation of the equation of the conicoid in Ex. 21 in the case when $\alpha=l'$.

23. Prove that if the joins of the mid-points of AB, CD; AC, DB; AD, BC are taken as coordinate axes, the equation to any conicoid through the four sides of the skew quadrilateral ABCD is of the form

$$\left(\frac{x}{a}+\frac{y}{b}\right)^2-\left(\frac{z}{c}-1\right)^2=\lambda\left\{\left(\frac{x}{a}-\frac{y}{b}\right)^2-\left(\frac{z}{c}+1\right)^2\right\},$$

where λ is a parameter. What surfaces correspond to (i) $\lambda=1$, (ii) $\lambda=-1$?

24. Find the locus of the centres of hyperboloids of one sheet that pass through the sides of a given skew quadrilateral.

25. If a conicoid passes through the edges AB, BC, CD of a tetrahedron, the pole of the plane bisecting the edges AB, CD, AC, BD will lie on the plane bisecting the edges AB, CD, AD, BC.

26. If the intersection of two conicoids consists of a conic and two straight lines through a point P of the conic, the sections of the conicoids by any plane through P have contact of the second order unless the plane passes through the tangent to the conic at P, when the contact is of the third order.

27. A cone, vertex P, and a conicoid S have two plane sections common. The conicoids S_1 and S_2 each touch S along one of the curves of section. Prove that if S_1 and S_2 pass through P, they touch at P and have a common conic section lying in the polar plane of P with respect to S.

28. If three cones C_1, C_2, C_3 have their vertices collinear and C_1, C_2; C_2, C_3 intersect in plane curves, then C_3, C_1 intersect also in plane curves and the six planes of intersection pass through one line.

29. If conicoids pass through the curve of intersection of a given conicoid and a given sphere whose centre is O, the normals to them from O lie on a cone of the second degree, and the feet of the normals lie on a curve of the third degree which is the locus of the centres of the conicoids.

30. Two conicoids are inscribed in the same cone and any secant through the vertex meets them in P, P′; Q, Q′. Prove that the lines of intersection of the tangent planes at P, Q; P, Q′; P′, Q; P′, Q′ lie in one of two fixed planes.

31. The sides of a skew quadrilateral ABCD are along generators of a hyperboloid, and any transversal meets the hyperboloid in P_1, P_2 and the planes ABC, BCD, CDA, DAB in A_1, A_2, B_1, B_2. Prove that

$$\frac{P_1A_1 . P_1B_1}{P_1A_2 . P_1B_2}=\frac{P_2A_1 . P_2B_1}{P_2A_2 . P_2B_2}.$$

32. A curve is drawn on the sphere $x^2+y^2+z^2=a^2$ so that at any point the latitude is equal to the longitude. Prove that it also lies on the cylinder $x^2+y^2=ax$. Shew that the curve is a quartic curve, that its equations may be written

$$x=\frac{a(1-t^2)^2}{(1+t^2)^2}\quad y=\frac{2at(1-t^2)}{(1+t^2)^2},\quad z=\frac{2at}{1+t^2},$$

and that if t_1, t_2, t_3, t_4 are the values of t for the four points in which the curve meets any given plane, $t_1t_2t_3t_4=1$.

33. The general equation to a conicoid through the feet of the normals from a point to an ellipsoid, $S=0$, is

$$S+\lambda C_1+\mu C_2+\nu C_3=0,$$

where $C_1=0$ $C_2=0$, $C_3=0$ represent cylinders through the feet of the normals.

Prove that the axes of paraboloids of revolution that pass through the feet of six concurrent normals to the conicoid $ax^2+by^2+cz^2=1$ are parallel to one of the lines

$$\frac{x^2}{-a+b+c}=\frac{y^2}{a-b+c}=\frac{z^2}{a+b-c}.$$

34. Prove that the cone whose vertex is $(a, 0, 0)$ and base

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\quad x=0,$$

intersects the cone whose vertex is $(0, b, 0)$ and base

$$\frac{z^2}{c^2}+\frac{x^2}{a^2}=1,\quad y=0,$$

in a parabola of latus rectum $\dfrac{2c^2}{\sqrt{a^2+b^2}}$.

CHAPTER XIII.

THE CONOIDS.

175. A cone is the surface generated by a straight line which passes through a fixed point and intersects a given curve, and a cylinder is the surface generated by the parallels to a given straight line which intersect a given curve. These are the most familiar of the ruled surfaces. Another important class of ruled surfaces, **the conoids**, may be defined as follows: a conoid is the locus of a line which always intersects a fixed line and a given curve and is parallel to a given plane. If the given line is at right angles to the given plane, the locus is a right conoid.

Ex. The hyperbolic paraboloid is a conoid, since it is the locus of a line which intersects two given lines and is parallel to a given plane, (§ 50, Ex. 3).

176. The equation to a conoid. If the coordinate axes be chosen so that the given line is the z-axis and the given plane the xy-plane, the generators of the conoid will project the given curve on the plane $x=1$ in a curve whose equation, let us suppose, is $z=f(y)$. Let P, $(1, y_1, z_1)$, be any point of this curve; then $z_1=f(y_1)$. The generator of the conoid through P is the line joining P to the point $(0, 0, z_1)$, and therefore has equations

$$\frac{x}{1}=\frac{y}{y_1}=\frac{z-z_1}{0}.$$

Eliminating y_1 and z_1 between these equations and the equation $z_1=f(y_1)$, we obtain the equation to the conoid, viz.,

$$z=f(y/x).$$

Ex. 1. Find the equation to the right conoid generated by lines which meet OZ, are parallel to the plane XOY, and intersect the circle $x=a$, $y^2+z^2=r^2$. *Ans.* $x^2(z^2-r^2)+a^2y^2=0$.

Ex. 2. The graph of $c\sin\theta$, from $\theta=0$ to $\theta=2r\pi$, is wrapped round the cylinder $x^2+y^2=r^2$ so that the extremities of the graph coincide on **OX**. Lines parallel to the plane **XOY** are drawn to meet **OZ** and the curve so formed. Prove that the equation to the conoid they generate is

$$r\tan^{-1}\frac{y}{x}=\sin^{-1}\frac{z}{c}.$$

Ex. 3. Prove that if $r=2$, the equation to the locus becomes

$$z(x^2+y^2)=2cxy.$$

(The locus is **the cylindroid.**)

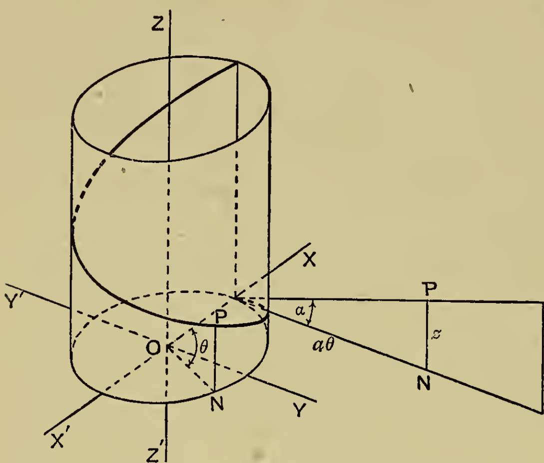

Fig. 51.

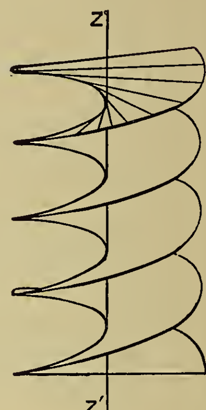

Fig. 52.

Ex. 4. The curve drawn on the right cylinder $x^2+y^2=a^2$ so as to cut all the generators at the same angle is called the right circular **helix.** The coordinates of any point on it are easily seen, (fig. 51), to be given by

$$x=a\cos\theta,\quad y=a\sin\theta,\quad z=a\theta\tan\alpha.$$

The conoid generated by lines parallel to the plane **XOY** which intersect the z-axis and the helix is the **helicoid,** (fig. 52). Shew that its equation is

$$z=c\tan^{-1}y/x,\ \text{where } c\equiv a\tan\alpha.$$

Ex. 5. Lines parallel to the plane **XOY** are drawn to intersect **OZ** and the curves

$$\text{(i)}\ x^2+y^2=r^2,\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c};$$

$$\text{(ii)}\ x^2+y^2+z^2=b^2,\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Find the equation to the conoids generated.

Ans. (i) $cr^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=2z(x^2+y^2)$,

(ii) $(b^2-z^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=\left(1-\frac{z^2}{c^2}\right)(x^2+y^2)$.

Ex. 6. Discuss the form of the conoids represented by

$$\text{(i) } y^2z=4acx, \quad \text{(ii) } yz^3=ax^3.$$

Ex. 7. Conoids are constructed as in Ex. 2 with the graphs of $c\operatorname{cosec}\theta$, $c\tan\theta$. Find their equations, considering specially the cases in which $r=1$ and $r=2$.

Ans. $r\tan^{-1}\dfrac{y}{x}=\operatorname{cosec}^{-1}\dfrac{z}{c}$, $\quad r\tan^{-1}\dfrac{y}{x}=\tan^{-1}\dfrac{z}{c}$.

$$r=1, \quad c^2(x^2+y^2)=y^2z^2, \quad xz=cy\,;$$
$$r=2, \quad c(x^2+y^2)=2xyz, \quad 2cxy=z(x^2-y^2).$$

Ex. 8. A curve is drawn on a right cone, semi-vertical angle α, so as to cut all the generators at the same angle, β, and a right conoid is generated by lines which meet the curve and cut OZ at right angles. Prove that the coordinates of any point on it are given by

$$x=u\cos\theta, \quad y=u\sin\theta, \quad z=ae^{m\theta},$$

where $$m\equiv\sin\alpha\cot\beta.$$

SURFACES IN GENERAL.

177. We shall now obtain some general properties of surfaces which are represented by an equation in cartesian coordinates. In the following paragraphs it will sometimes be convenient to use ξ, η, ζ to denote current coordinates.

The general equation of the n^{th} degree may be written

$$u_0+u_1+u_2+\ldots+u_n=0,$$

where u_r stands for the general homogeneous expression in x, y, z of degree r. The number of terms in u_r is

$$\frac{(r+1)(r+2)}{1.2},$$

and therefore the number of terms in the general equation is

$$\sum_{r=0}^{r=n}\frac{(r+1)(r+2)}{1.2}, \text{ or } \frac{(n+1)(n+2)(n+3)}{1.2.3}\equiv \mathsf{N}+1, \text{ say.}$$

Hence the equation contains N disposable constants, and a surface represented by an equation of the n^{th} degree can be found to satisfy N conditions which each involve one relation between the constants.

Ex. 1. In the general cubic equation there are 19 disposable constants, and a surface represented by a cubic equation can be found to pass through 19 given points.

Ex. 2. A cubic surface contains 27 straight lines, real or imaginary.

If $u=0$, $v=0$, $w=0$, $u_1=0$, $v_1=0$, $w_1=0$ represent arbitrary planes, the equation

$$uvw+\lambda u_1v_1w_1=0$$

contains 19 disposable constants, and therefore can be identified with any cubic equation. Suppose then that the equation to the given surface has been thrown into this form. Clearly the lines

$$\begin{array}{lll} u=0,\quad u_1=0; & u=0,\quad v_1=0; & u=0,\quad w_1=0; \\ v=0,\quad u_1=0; & v=0,\quad v_1=0; & v=0,\quad w_1=0; \\ w=0,\quad u_1=0; & w=0,\quad v_1=0; & w=0,\quad w_1=0 \end{array}$$

lie upon the surface, so that the surface contains at least nine straight lines, real or imaginary

Consider now the equation

$$uv=\kappa u_1v_1$$

It represents a hyperboloid of one sheet which intersects the surface at points which lie in the plane $\kappa w+\lambda w_1=0$. Now κ can be chosen so that this plane is a tangent plane to the hyperboloid, and then the common points lie upon the two generators of the hyperboloid which are in the plane. Thus the surface contains two other straight lines. But since each of the sets of quantities u, v, w; u_1, v_1, w_1 can be divided into groups of two in three ways, there are nine hyperboloids, each of which has two generators lying upon the surface.

The surface therefore contains twenty-seven straight lines, real or imaginary.

178. The degree of a surface. If an arbitrary straight line meets a surface in n points the surface is of the n^{th} degree.

Consider the surface represented by the equation of the n^{th} degree, $\mathsf{F}(\xi, \eta, \zeta)=0$. The straight line whose equations are

$$\frac{\xi-x}{l}=\frac{\eta-y}{m}=\frac{\zeta-z}{n} \quad (=\rho)$$

meets the surface at points whose distances from (x, y, z) are given by

$$\mathsf{F}(x+l\rho,\ y+m\rho,\ z+n\rho)=0,$$

i.e.

$$\mathsf{F}(x, y, z)+\rho\left(l\frac{\partial}{\partial x}+m\frac{\partial}{\partial y}+n\frac{\partial}{\partial z}\right)\mathsf{F}$$

$$+\frac{\rho^2}{\underline{|2}}\left(l\frac{\partial}{\partial x}+m\frac{\partial}{\partial y}+n\frac{\partial}{\partial z}\right)^2\mathsf{F}+\dots$$

$$+\frac{\rho^n}{\underline{|n}}\left(l\frac{\partial}{\partial x}+m\frac{\partial}{\partial y}+n\frac{\partial}{\partial z}\right)^n\mathsf{F}=0. \quad \dots\dots\dots\dots(1)$$

This equation gives n values of ρ, and therefore the line

meets the surface in n points. Hence the locus of an equation of the n^{th} degree is a surface of the n^{th} degree.

Cor. Any plane section of a surface of the n^{th} degree is a curve of the n^{th} degree.

179. Tangents and tangent planes. If in equation (1) of the last paragraph, $F(x, y, z)=0$, the point (x, y, z) is on the surface. If also

$$l\frac{\partial F}{\partial x}+m\frac{\partial F}{\partial y}+n\frac{\partial F}{\partial z}=0, \quad \text{.....................}(2)$$

the equation gives two zero values of ρ, and the line meets the surface at (x, y, z) in two coincident points. If therefore

$$\frac{\partial F}{\partial x}, \quad \frac{\partial F}{\partial y}, \quad \frac{\partial F}{\partial z}$$

are not all zero, the system of lines whose direction-ratios satisfy equation (2) touches the surface at (x, y, z), and the locus of the system is the tangent plane at (x, y, z), which is given by

$$(\xi-x)\frac{\partial F}{\partial x}+(\eta-y)\frac{\partial F}{\partial y}+(\zeta-z)\frac{\partial F}{\partial z}=0.$$

If the equation to the surface is made homogeneous by the introduction of an auxiliary variable t which is equated to unity after differentiation, the equation to the tangent plane may be reduced, as in § 134, to the form

$$\xi\frac{\partial F}{\partial x}+\eta\frac{\partial F}{\partial y}+\zeta\frac{\partial F}{\partial z}+t\frac{\partial F}{\partial t}=0.$$

Ex. 1. Find the equation to the tangent plane at a point (x, y, z) of the surface $\xi\eta\zeta=a^3$. *Ans.* $\xi/x+\eta/y+\zeta/z=3$.

Ex. 2. The feet of the normals from a given point to the cylindroid

$$z(x^2+y^2)=2cxy$$

lie upon a conicoid.

180. The inflexional tangents. Two values of the ratios $l:m$, $l:n$ can be found to satisfy the equations

$$l\frac{\partial F}{\partial x}+m\frac{\partial F}{\partial y}+n\frac{\partial F}{\partial z}=0,$$

$$l^2\frac{\partial^2 F}{\partial x^2}+m^2\frac{\partial^2 F}{\partial y^2}+n^2\frac{\partial^2 F}{\partial z^2}+2mn\frac{\partial^2 F}{\partial y\partial z}+2nl\frac{\partial^2 F}{\partial z\partial x}+2lm\frac{\partial^2 F}{\partial x\partial y}=0,$$

formed by equating to zero the coefficients of ρ and ρ^2 in equation (1) of § 178. The lines through (x, y, z) whose directions are determined by these values meet the surface in three coincident points. That is, in the system of tangent lines through (x, y, z) there are two which have contact of higher order than the others. They are called the **inflexional tangents** at (x, y, z). They may be real and distinct, as in the hyperboloid of one sheet, real and coincident, as in a cone or cylinder, or imaginary, as in the ellipsoid.

The section of the surface by the tangent plane at a point P on it is a curve of the n^{th} degree, and any line through P which lies in the tangent plane meets the curve in two coincident points. P is therefore a double point of the curve. The inflexional tangents at P meet the curve in three coincident points, and are therefore the tangents to the curve at the double point. Hence, if the inflexional tangents through P are real and distinct, P is a node on the curve; if they are real and coincident, P is a cusp; if they are imaginary, P is a conjugate point.

181. The equation $\zeta=f(\xi, \eta)$. If the equation to the surface is given in the form $\zeta=f(\xi, \eta)$, the values of ρ corresponding to the points of intersection of the surface and the line

$$\frac{\xi-x}{l}=\frac{\eta-y}{m}=\frac{\zeta-z}{n} \quad (=\rho)$$

are given by

$$z+n\rho=f(x+l\rho,\ y+m\rho),$$
$$=f(x, y)+\rho(pl+qm)+\frac{\rho^2}{\lfloor 2}(rl^2+2slm+tm^2)+\ldots,$$

where $p\equiv\dfrac{\partial z}{\partial x}$, $q\equiv\dfrac{\partial z}{\partial y}$, $r\equiv\dfrac{\partial^2 z}{\partial x^2}$, $s\equiv\dfrac{\partial^2 z}{\partial x\partial y}$, $t\equiv\dfrac{\partial^2 z}{\partial y^2}$.

Hence the tangent plane at (x, y, z) has for its equation

$$p(\xi-x)+q(\eta-y)-(\zeta-z)=0,$$

and the inflexional tangents are the lines of intersection of the tangent plane and the pair of planes given by

$$r(\xi-x)^2+2s(\xi-x)(\eta-y)+t(\eta-y)^2=0.$$

Ex. 1. The inflexional tangents through any point of a conoid are real.

One inflexional tangent is the generator through the point, and is therefore real. Hence the other must also be real.

Or thus: The inflexional tangents are real, coincident, or imaginary according as $rt-s^2 \lesseqgtr 0$.

For the conoid $z=f(y/x)$,

$$p=-\frac{y}{x^2}f', \quad q=\frac{1}{x}f',$$

$$r=\frac{2y}{x^3}f'+\frac{y^2}{x^4}f'', \quad s=-\frac{1}{x^2}f'-\frac{y}{x^3}f'', \quad t=\frac{1}{x^2}f'',$$

and hence $\quad rt-s^2=-\dfrac{1}{x^4}f'^2.$

Ex. 2. Find the equations to the inflexional tangents through a point (x, y, z) of the surface (i) $\eta^2\zeta=4c\xi$, (ii) $\zeta^3\eta=a\xi^3$.

Ans. (i) $\dfrac{\xi-x}{3x}=\dfrac{\eta-y}{2y}=\dfrac{\zeta-z}{-z}, \quad \dfrac{\xi-x}{y^2}=\dfrac{\eta-y}{0}=\dfrac{\zeta-z}{4c};$

(ii) $\eta=y, \quad a\xi x^2-\zeta y z^2=0, \quad \dfrac{\xi-x}{2x}=\dfrac{\eta-y}{3y}=\dfrac{\zeta-z}{z}.$

Ex. 3. Any point on the cylindroid

$$z(x^2+y^2)=2cxy$$

is given by $\quad x=u\cos\theta, \quad y=u\sin\theta, \quad z=c\sin 2\theta.$

Prove that the inflexional tangents through "u, θ" have for equations

$$\frac{x-u\cos\theta}{-u\sin 3\theta}=\frac{y-u\sin\theta}{u\cos 3\theta}=\frac{z-c\sin 2\theta}{2c\cos^2 2\theta},$$

$$\frac{x}{\cos\theta}=\frac{y}{\sin\theta}=\frac{z-c\sin 2\theta}{0}.$$

Ex. 4. Find the locus of points on the cylindroid at which the inflexional tangents are at right angles.

182. Singular points. If at a point P, (x, y, z), of the surface

$$\frac{\partial F}{\partial x}=\frac{\partial F}{\partial y}=\frac{\partial F}{\partial z}=0,$$

every line through P meets the surface in two coincident points. P is then a **singular point** of the first order. The lines through P whose direction-ratios satisfy the equation

$$\left(l\frac{\partial}{\partial x}+m\frac{\partial}{\partial y}+n\frac{\partial}{\partial z}\right)^2 F=0$$

meet the surface in three coincident points at P, and are

the tangents at the singular point. The locus of the system of tangents through P is the surface

$$(\xi-x)^2\frac{\partial^2 F}{\partial x^2}+\ldots 2(\eta-y)(\zeta-z)\frac{\partial^2 F}{\partial y\partial z}+\ldots=0.$$

Singular points are classified according to the nature of the locus of the tangent lines. When the locus is a proper cone, P is a **conical point** or **conic node**, when it is a pair of distinct planes, P is a **biplanar node** or **binode**, when the biplanes coincide, P is a **uniplanar node** or **unode.**

The six tangents through a singular point P, (x, y, z), whose direction-ratios satisfy the equations

$$\left(l\frac{\partial}{\partial x}+m\frac{\partial}{\partial y}+n\frac{\partial}{\partial z}\right)^2 F=0,\quad \left(l\frac{\partial}{\partial x}+m\frac{\partial}{\partial y}+n\frac{\partial}{\partial z}\right)^3 F=0,$$

have four-point contact with the surface at P. They correspond to the inflexional tangents at an ordinary point of the surface.

Ex. 1. For the surface

$$x^4+y^4+z^4+6xyz+2x^2-y^2+z^2+4yz+3zx=0,$$

the origin is a conic node. The locus of the tangents at the origin is the cone

$$2x^2-y^2+z^2+4yz+3zx=0.$$

The six tangents which have four-point contact are

$$x=0,\ y=(2\pm\sqrt{5})z;\quad y=0,\ 2x+z=0;$$
$$y=0,\ x+z=0;\quad z=0,\ \sqrt{2}x=\pm y.$$

Ex. 2. For the surface

$$x^4+y^4+z^4+3xyz+x^2-2y^2-3z^2-5yz+2zx+xy=0,$$

the origin is a binode. The six tangents with four-point contact are

$$x=0,\ 2y+3z=0;\quad x=0,\ y+z=0;\quad y=0,\ 3z+x=0;$$
$$y=0,\ z-x=0;\quad z=0,\ x+2y=0;\quad z=0,\ x-y=0.$$

The sections of the surface by the planes $x+2y+3z=0$, $x-y-z=0$, have a triple point at the origin.

Ex. 3. The equation to a surface is of the form

$$z^2+u_3+u_4+\ldots+u_n=0.$$

Prove that there is a unode at the origin, that the section of the surface by the plane $z=0$ has a triple point at the origin, and that the three tangents there, counted twice, are the tangents to the surface with four-point contact.

Ex. 4. The equation to a surface is of the form

$$x^2+y^2+u_3+v_4+\ldots+u_n=0.$$

Shew that the section of the surface by any plane through OZ has a cusp at the origin.

Ex. 5. For the surface

$$xy+z(ax^2+2hxy+by^2)+z^2(cx+dy)=0,$$

prove that the origin is a binode and that the line of intersection of the biplanes lies on the surface. Shew that the plane $cx+dy=0$ is a tangent plane at any point of OZ.

Ex. 6. Find and classify the singular points of the surfaces

$$\text{(i)}\ a^2x^2-b^2y^2=z^3(c-z),$$
$$\text{(ii)}\ xyz=ax^2+by^2+cz^2,$$
$$\text{(iii)}\ x(x^2+3y^2+3z^2)=3a(x^2-y^2-z^2),$$
$$\text{(iv)}\ xyz-a^2(x+y+z)+2a^3=0.$$

Ans. (i) (0, 0, 0) is binode; (ii) (0, 0, 0) is conic node; (iii) (0, 0, 0) is conic node, (the surface is formed by revolution of the curve $x(x^2+3y^2)=3a(x^2-y^2)$, $z=0$, about OX); (iv) (a, a, a) is a conic node.

Ex. 7. Find the equation to the surface generated by a variable circle passing through the points $(0, 0, \pm c)$ and intersecting the circle $z=0$, $x^2+y^2=2ax$, and shew that the tangent cones at the conical points intersect the plane $z=0$ in the conic

$$(c^2-4a^2)x^2+c^2y^2=4ac^2x.$$

Ex. 8. If every point of a line drawn on a surface is a singular point, the line is a **nodal line**. Find the nodal lines of the surfaces

$$\text{(i)}\ z(x^2+y^2)=2axy,$$
$$\text{(ii)}\ c^2(x^2+y^2)^2=a^2z^2(x^2-y^2),$$
$$\text{(iii)}\ (y^2+z^2)\{(2x-y)^2+z^2\}=4a^2z^2.$$

Ans. (i) $x=y=0$; (ii) $x=y=0$; (iii) $y=z=0$, $y-2x=z=0$.

Ex. 9. Prove that the z-axis is a nodal line on the surface

$$2xy+ax^3+3bx^2y+3cxy^2+dy^3+z(px^2+2qxy+ry^2)=0,$$

any point $(0, 0, \gamma)$ being a binode at which the tangent planes are

$$2xy+\gamma(px^2+2qxy+ry^2)=0.$$

Prove also that if r and p have the same sign there are two real unodes lying on the nodal line.

Ex. 10. For the surface

$$2xy+x^3-3x^2y-3xy^2+y^3+z(x^2-xy+y^2)=0,$$

prove that the z-axis is a nodal line with unodes at the points $(0, 0, -2)$, $(0, 0, \frac{2}{3})$.

183. Singular tangent planes. We have seen that the tangent plane at a point P of a surface meets the surface in a curve which has a double point at P. The curve may

have other double points. If Q is another double point, the plane contains the inflexional tangents to the surface at Q, and is therefore the tangent plane at Q. A plane which is a tangent plane at two points of a surface is a **double tangent plane.** We may likewise have planes touching at three points of the surface or **triple tangent planes**, and tangent planes touching at four or more points of the surface.

Or we may have a tangent plane which touches the surface at all points of a curve, as the tangent plane to a cone or cylinder. Such a plane is a **singular tangent plane** or **trope.***

Ex. 1. For the cubic surface $uvw+u_1v_1w_1=0$, the planes $u=0$, $v=0$, $w=0$, $u_1=0$, $v_1=0$, $w_1=0$ are triple tangent planes.

The intersection of the plane $u=0$ and the surface is the cubic curve consisting of the three straight lines $u=u_1=0$, $u=v_1=0$, $u=w_1=0$. These lines form a triangle and the three vertices are double points, so that the plane $u=0$ is tangent plane at three points.

Ex. 2. Find the singular point on the surface

$$(x^2+y^2+z^2)^2=4a^2(x^2+y^2),$$

and shew that the planes $z=\pm a$ are singular tangent planes.

Ex. 3. Sketch the form of the cone

$$c^2(x^2+y^2)^2=a^2z^2(x^2-y^2),$$

and shew that the planes $2\sqrt{2}cy=\pm az$ each touch it along two generators.

The sections by planes parallel to XOY are lemniscates.

Ex. 4. Prove that the planes $z=\pm c$ are singular tangent planes to the cylindroid $z(x^2+y^2)=2cxy$.

THE ANCHOR-RING.

Ex. 5. The surface generated by the revolution of a circle about a line in its plane which it does not intersect is called the **anchor-ring** or **tore.**

If the straight line is the z-axis and the circle is $y=0$, $(x-a)^2+z^2=b^2$, $(a>b)$, shew that the equation to the surface is

$$(x^2+y^2+z^2+a^2-b^2)^2=4a^2(x^2+y^2).$$

Prove that the planes $z=\pm b$ are singular tangent planes.

---

*For an adequate discussion of the singularities of surfaces the student is referred to Basset's *Geometry of Surfaces.* An interesting account of the properties of cubic surfaces with methods for the construction of models is given in *Cubic Surfaces*, by W. H. Blythe. *Kummer's Quartic Surface* (Hudson) contains an exposition of the properties of various quartic surfaces.

Ex. 6. Prove that the polar equation of the curve of intersection of the surface and the tangent plane $x=a-b$, referred to a line parallel to OY as initial line, is $r^2=4a^2\sin(\alpha-\theta)\sin(\alpha+\theta)$, where $\sin\alpha=\sqrt{b/a}$.

Ex. 7. Prove that the inflexional tangents at $(a-b,\ 0,\ 0)$ are $x=a-b$, $y\sqrt{b}=\pm z\sqrt{a-b}$.

Ex. 8. The tangent plane which passes through OY is $z=x\tan\alpha$, where $\sin\alpha=b/a$ and it touches the surface at the two points

$$(a\cos^2\alpha,\ 0,\ a\cos\alpha\sin\alpha),\quad(-a\cos^2\alpha,\ 0,\ -a\cos\alpha\sin\alpha).$$

Where it meets the surface we have

$$x\sin\alpha=z\cos\alpha,\quad(x^2+y^2+z^2+a^2-b^2)^2=4a^2(x^2+y^2)\ ;$$

therefore

$$\begin{aligned}(x^2+y^2+z^2-a^2\cos^2\alpha)^2&=4a^2(x^2+y^2)-4a^2(x^2+y^2+z^2)\cos^2\alpha,\\&=4a^2(x^2+y^2)\sin^2\alpha-4a^2z^2\cos^2\alpha,\\&=4a^2y^2\sin^2\alpha.\end{aligned}$$

Hence $$x^2+(y\pm a\sin\alpha)^2+z^2=a^2.$$

Therefore the curve of intersection of the surface and the tangent plane consists of two circles which intersect at the points of contact

$$(a\cos^2\alpha,\ 0,\ a\cos\alpha\sin\alpha),\quad(-a\cos^2\alpha,\ 0,\ -a\cos\alpha\sin\alpha).$$

THE WAVE SURFACE.

If N′ON is normal to any central section of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

and lengths OA, OA′ ; OB, OB′ equal to the axes of the section are measured along ON and ON′, the points A, A′, B, B′ lie upon a surface of the fourth degree, which is called the **wave surface.** Since the axes of the section by the plane $lx+my+nz=0$ are given by

$$\frac{a^2l^2}{a^2-r^2}+\frac{b^2m^2}{b^2-r^2}+\frac{c^2n^2}{c^2-r^2}=0,$$

the equation to the wave surface is

$$\frac{a^2x^2}{a^2-r^2}+\frac{b^2y^2}{b^2-r^2}+\frac{c^2z^2}{c^2-r^2}=0,$$

where $r^2\equiv x^2+y^2+z^2$. The equation, on simplification, becomes

$$(x^2+y^2+z^2)(a^2x^2+b^2y^2+c^2z^2)-a^2(b^2+c^2)x^2-b^2(c^2+a^2)y^2-c^2(a^2+b^2)z^2\\+a^2b^2c^2=0.$$

If the plane of section of the ellipsoid passes through one of the principal axes, that axis is an axis of the conic in which the plane cuts the ellipsoid. Thus one of the axes of any section through Y′OY is equal to b. The remaining axes of such sections coincide in turn with the semi-diameters of the ellipse $y=0$, $x^2/a^2+z^2/c^2=1$. Hence the points A, A′, B, B′, corresponding to sections through Y′OY, describe a circle of radius b and an ellipse which is simply the above ellipse turned through a right angle, and whose equations are therefore $y=0$,

$x^2/c^2+z^2/a^2=1$. The circle and this ellipse clearly form the intersection of the wave surface and the plane $y=0$.

The result can be immediately verified by putting y equal to zero in the equation to the surface, when we obtain

$$(z^2+x^2-b^2)(c^2z^2+a^2x^2-c^2a^2)=0.$$

Similarly, the sections of the surface by the planes $x=0$, $z=0$ are the circles and ellipses given by

$$x=0, \quad (y^2+z^2-a^2)(b^2y^2+c^2z^2-b^2c^2)=0\,;$$

$$z=0, \quad (x^2+y^2-c^2)(a^2x^2+b^2y^2-a^2b^2)=0.$$

Fig. 53 shews an octant of the wave surface.

If $a>b>c$, the only two of these circles and ellipses which have common points lie in the plane $y=0$, and the points are given by

$$\frac{ax}{\sqrt{a^2-b^2}}=\frac{y}{0}=\frac{cz}{\sqrt{b^2-c^2}}=\pm\frac{ac}{\sqrt{a^2-c^2}}.$$

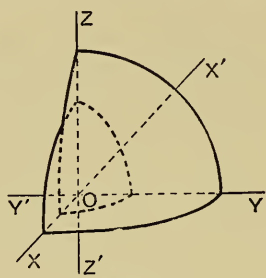

FIG. 53.

The wave surface consists of two sheets, one described by points such as A and A′, the other by points such as B and B′. The sheets will cross only where the axes of the central sections are equal. Hence since there are only two real central circular sections, and the radius of each is b, the only four points common to the two sheets lie on the normals to the central circular sections, and are at a distance b from the centre. They are given by

$$\frac{a\xi}{\sqrt{a^2-b^2}}=\frac{\eta}{0}=\frac{c\zeta}{\sqrt{b^2-c^2}}=\pm\frac{b}{\sqrt{\dfrac{a^2-b^2}{a^2}+\dfrac{b^2-c^2}{c^2}}}=\frac{\pm ac}{\sqrt{a^2-c^2}},$$

and are thus the points of intersection of the circle and ellipse in the plane $y=0$, as clearly should be the case.

If P is one of these four points, the section of the surface by the plane $y=0$ has a double point at P, and the plane $y=0$ is not a tangent plane at P. This suggests that P is a singular point on the wave surface. Change the origin to P, (ξ, η, ζ), noting that

$$\xi^2+\zeta^2=b^2, \quad \eta=0, \quad a^2\xi^2+c^2\zeta^2=a^2c^2.$$

The equation becomes

$$4(a^2x\xi+c^2z\zeta)(x\xi+z\zeta)-y^2(a^2-b^2)(b^2-c^2)+\ldots=0,$$

and hence **P** is a conical point. Thus the wave surface has four conical points, and they are the points of intersection of the circle and ellipse which form the section of the surface by the plane $y=0$.

Since any plane section of the surface is a curve of the fourth degree, if the surface has a singular tangent plane, the intersection of the tangent plane with the surface will consist of two coincident conics, or the plane will touch the surface at all points of a conic. Any plane will meet the conic in two points **Q** and **R**, the singular tangent plane in the line **QR**, and the surface in a curve of the fourth degree which **QR** touches at **Q** and **R**. Considering, then, the sections of the surface by the coordinate planes, we see that any real singular tangent plane must pass through a common tangent to the circle and ellipse in the plane $y=0$. Their equations are

$$y=0,\ z^2+x^2=b^2\ ; \quad y=0,\ c^2z^2+a^2x^2=c^2a^2,$$

and the common tangents are easily found to be given by

$$y=0, \quad \pm\sqrt{a^2-b^2}x\pm\sqrt{b^2-c^2}z=b\sqrt{a^2-c^2},$$

or by
$$ax\xi+cz\zeta=abc, \quad y=0,$$

where (ξ, η, ζ) is one of the singular points.

If the equation to the surface is $f(x, y, z)=0$, $\dfrac{\partial f}{\partial y}=0$ when $y=0$, and hence the tangent plane at any point of the zx-plane is parallel to **OY**, and therefore the plane

$$ax\xi+cz\zeta=abc$$

is at least a double tangent plane. Now the equation to the surface can be written in the form

$$b^2(r^2-a^2)(r^2-c^2)+(a^2-b^2)(r^2-c^2)x^2-(b^2-c^2)(r^2-a^2)z^2=0,$$

or
$$\left\{b(r^2-a^2)+\frac{x\xi}{c}(a^2-c^2)\right\}\left\{b\,(r^2-c^2)-\frac{z\zeta}{a}(a^2-c^2)\right\}$$
$$+\frac{a^2-c^2}{ac}\left\{(r^2-c^2)\frac{x\xi}{c}-(r^2-a^2)\frac{z\zeta}{a}\right\}\{ax\xi+cz\zeta-abc\}=0.$$

Therefore the plane $ax\xi+cz\zeta=abc$ meets the surface at points lying on one of the spheres

$$b(r^2-a^2)+\frac{x\xi}{c}(a^2-c^2)=0, \quad b\,(r^2-c^2)-\frac{z\zeta}{a}(a^2-c^2)=0.$$

But, subtracting, we see that the common points of these spheres lie in the plane

$$\frac{x\xi}{c}+\frac{z\zeta}{a}=b,$$

or
$$ax\xi+cz\zeta=abc.$$

Thus the plane meets both spheres in the same circle, or the section of the surface by the plane consists of two coincident circles, and therefore the plane is a singular tangent plane. The wave surface has therefore four singular tangent planes.

184. The indicatrix. If the tangent plane and normal at a given point of a surface be taken as the plane $z=0$ and the z-axis, and the equation to the surface is then $z=f(x, y)$, this equation may be written

$$z=px+qy+\tfrac{1}{2}(rx^2+2sxy+ty^2)+\dots,$$

where p, q, r, s, t are the values of

$$\frac{\partial z}{\partial x}, \quad \frac{\partial z}{\partial y}, \quad \frac{\partial^2 z}{\partial x^2}, \quad \frac{\partial^2 z}{\partial x \partial y}, \quad \frac{\partial^2 z}{\partial y^2}$$

at the origin; or, since $p=q=0$,

$$2z=rx^2+2sxy+ty^2+\dots.$$

Hence, if we consider x and y in the neighbourhood of the origin to be small quantities of the first order, z is of the second order, and therefore, if we reject terms of the third and higher orders, we have as an approximation to the shape of the surface at the origin the conicoid given by

$$2z=rx^2+2sxy+ty^2.$$

This conicoid is a paraboloid if $rt \neq s^2$, and a parabolic cylinder if $rt=s^2$. In the neighbourhood of the origin the sections of the surface and conicoid by a plane parallel to the tangent plane, and at an infinitesimal distance h from it, coincide; the section of the conicoid is the conic given by

$$z=h, \quad 2h=rx^2+2sxy+ty^2,$$

which is called the **indicatrix**. The inflexional tangents are given by

$$z=0, \quad rx^2+2sxy+ty^2=0,$$

and are clearly parallel to the asymptotes of the indicatrix. Hence if the inflexional tangents are imaginary, the indicatrix is an ellipse, and the origin is an **elliptic point** on the surface; if they are real and distinct, the indicatrix is a hyperbola, and the origin is a **hyperbolic point**; and if they are coincident, the indicatrix is two parallel straight lines, and the origin is a **parabolic point**.

At an elliptic point the shape of the surface is approximately that of an elliptic paraboloid, and therefore the surface lies on one side of the tangent plane at the point. It is said to be **synclastic** in this case. At a hyperbolic

point the shape is approximately that of a hyperbolic paraboloid, and the surface lies on both sides of the tangent plane. At such a point it is said to be anticlastic.

Ex. 1. Every point on a cone or cylinder is a parabolic point.

Ex. 2. Find the locus of the parabolic points on the surface $F(x, y, z, t)=0$.

The direction of the inflexional tangents through (x, y, z) are given by

$$lF_x+mF_y+nF_z=0,$$

$$l^2F_{xx}+m^2F_{yy}+n^2F_{zz}+2mnF_{yz}+2nlF_{zx}+2lmF_{xy}=0.$$

Hence the inflexional tangents coincide if

$$\begin{vmatrix} F_{xx}, & F_{xy}, & F_{xz}, & F_x \\ F_{yx}, & F_{yy}, & F_{yz}, & F_y \\ F_{zx}, & F_{zy}, & F_{zz}, & F_z \\ F_x, & F_y, & F_z, & 0 \end{vmatrix}=0. \qquad\ldots\ldots(1)$$

But F_x is a homogeneous function of x, y, z, t, of degree $(n-1)$, and therefore
$$xF_{xx}+yF_{xy}+zF_{xz}+tF_{xt}=(n-1)F_x, \text{ etc.,}$$
by means of which equation (1) can be reduced to

$$\begin{vmatrix} F_{xx}, & F_{xy}, & F_{xz}, & F_{xt} \\ F_{yx}, & F_{yy}, & F_{yz}, & F_{yt} \\ F_{zx}, & F_{zy}, & F_{zz}, & F_{zt} \\ F_{tx}, & F_{ty}, & F_{tz}, & F_{tt} \end{vmatrix}=0.$$

This equation determines a surface whose curve of intersection with the given surface is the required locus.

Ex. 3. Prove that the points of intersection of the surface
$$x^4+y^4+z^4=a^4$$
and the coordinate planes are parabolic points.

Ex. 4. Prove that the parabolic points of the cylindroid
$$z(x^2+y^2)=2cxy$$
lie upon the lines $x-y=0,\ z=c$; $x+y=0,\ z=-c$.

Ex. 5. Prove that the indicatrix at a point of the surface $z=f(x, y)$ is a rectangular hyperbola if $(1+p^2)t+(1+q^2)r-2pqs=0$.

Ex. 6. Prove that the indicatrix at every point of the helicoid $\dfrac{z}{c}=\tan^{-1}\dfrac{y}{x}$ is a rectangular hyperbola.

Ex. 7. The points of the surface $xyz-a(yz+zx+xy)=0$, at which the indicatrix is a rectangular hyperbola, lie on the cone
$$x^4(y+z)+y^4(z+x)+z^4(x+y)=0.$$

185. Representation by parameters. If x, y, z are functions of two parameters u and v and are given by the equations

$$x=f_1(u, v), \quad y=f_2(u, v), \quad z=f_3(u, v),$$

the locus of the point (x, y, z) is a surface. For u and v can be eliminated between the three equations, and the elimination leads to an equation of the form $\mathsf{F}(x, y, z)=0$.

The tangent plane. To find the equation to the tangent plane we may proceed thus. The equation is

$$(\xi-x)\mathsf{F}_x+(\eta-y)\mathsf{F}_y+(\zeta-z)\mathsf{F}_z=0.$$

But since x, y, z are functions of u and v,

$$\mathsf{F}_x x_u+\mathsf{F}_y y_u+\mathsf{F}_z z_u=0$$

and

$$\mathsf{F}_x x_v+\mathsf{F}_y y_v+\mathsf{F}_z z_v=0.$$

Therefore $$\frac{\mathsf{F}_x}{y_u z_v-z_u y_v}=\frac{\mathsf{F}_y}{z_u x_v-x_u z_v}=\frac{\mathsf{F}_z}{x_u y_v-y_u x_v}.$$

These give the direction-cosines of the normal.

The equation to the tangent plane is

$$\begin{vmatrix} \xi-x, & \eta-y, & \zeta-z \\ x_u, & y_u, & z_u \\ x_v, & y_v, & z_v \end{vmatrix}=0.$$

Ex. 1. Find the tangent plane at the point "u, θ" on the helicoid, for which

$$x=u\cos\theta, \quad y=u\sin\theta, \quad z=c\theta.$$

Ex. 2. Find the tangent plane at the point "u, θ" on the cylindroid, for which

$$x=u\cos\theta, \quad y=u\sin\theta, \quad z=c\sin 2\theta,$$

and prove that its intersection with the surface consists of a straight line and an ellipse whose projection on the plane $z=0$ is the circle

$$(x^2+y^2)\cos 2\theta-u(x\cos\theta-y\sin\theta)=0.$$

Ex. 3. Prove that the normals at points on the cylindroid for which θ is constant lie on a hyperbolic paraboloid.

Ex. 4. Prove that the equations

$$x=a_1\lambda+b_1\mu+c_1\lambda\mu, \quad y=a_2\lambda+b_2\mu+c_2\lambda\mu, \quad z=a_3\lambda+b_3\mu+c_3\lambda\mu$$

determine a hyperbolic paraboloid if $\Delta\neq 0$, and a pair of planes if $\Delta=0$, where

$$\Delta\equiv\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}.$$

Ex. 5. If $\Delta\neq 0$, prove that the equations

$$x=a_1\lambda^2+b_1\lambda\mu+c_1\mu^2, \quad y=a_2\lambda^2+b_2\lambda\mu+c_2\mu^2, \quad z=a_3\lambda^2+b_3\lambda\mu+c_3\mu^2$$

determine a cone whose vertex is the origin and which has as generators the lines

$$x/a_1=y/a_2=z/a_3, \quad x/c_1=y/c_2=z/c_3.$$

Examples X.

1. Prove that the surfaces

$$2xyz - x^2 - y^2 - z^2 + 1 = 0,$$
$$z(x^2 + y^2 - z^2 + 1) = 2xy$$

have each four conic nodes whose coordinates are

$$(1, 1, 1), \quad (1, -1, -1), \quad (-1, 1, -1), \quad (-1, -1, 1).$$

2. Prove that the surface

$$(x + y + z - a)^3 = xyz$$

has binodes at the points $(a, 0, 0)$, $(0, a, 0)$, $(0, 0, a)$.

3. Prove that the line $2x = a$, $z = 0$ is a nodal line on the surface

$$4cz^2(x - a) + by(2x - a)^2 = 0,$$

and that there is a unode at the point where it meets the plane $y = 0$. Prove also that the section of the surface by any plane through the nodal line consists of three straight lines, two of which coincide with the nodal line.

4. Prove that the surface

$$(x^2 + y^2)(3y - z)^2 = 4x^2$$

contains an infinite number of straight lines. Examine the nature of the sections by planes through the line $x = 3y - z = 0$.

5. Prove that the equation

$$a(y - b)(z - c)^2 - b(x - a)(z + c)^2 = 0$$

represents a conoid which is generated by lines parallel to the plane **XOY** which meet the line $x = a$, $y = b$. Shew also that the normals to the surface at points of the generator $x/a = y/b$, $z = 0$, lie on the hyperbolic paraboloid

$$4ab(bx - ay)(ax + by - a^2 - b^2) = cz(a^2 + b^2)^2.$$

6. Shew that the equation

$$x^3 + y^3 + z^3 - 3xyz = a^3$$

represents a surface of revolution, and find the equations to the generating curve.

7. Prove that the perpendiculars from the point (α, β, γ) to the generators of the cylindroid

$$x = u\cos\theta, \quad y = u\sin\theta, \quad z = c\sin 2\theta$$

lie on the conicoid

$$\gamma(x - \alpha)^2 + \gamma(y - \beta)^2 + 2c(x - \alpha)(y - \beta) - (z - \gamma)(\alpha x + \beta y - \alpha^2 - \beta^2) = 0.$$

8. Prove that the only real lines lying on the surface $x^3 + y^3 + z^3 = a^3$ are

$$x = a, \ y + z = 0\,; \quad y = a, \ z + x = 0\,; \quad z = a, \ x + y = 0.$$

Shew also that the section of the surface by a plane through one of these lines consists of a straight line and a conic. Determine the position of the plane through the line $x = a$, $y + z = 0$ which meets the surface in a conic whose projection on the yz-plane is a circle.

9. Shew that an infinite number of spheres with centres on the xy-plane cuts the surface $(x^2+y^2)(x+a)+z^2(x-a)=0$ at right angles, and find the locus of their centres.

10. Discuss the form of the surface

$$y^2z^2+2kxyz+k^2x^2-2ak^2y=0.$$

Shew that it is a ruled surface, and give a geometrical construction for the generator through a given point of the parabola in which it meets the xy-plane. Prove also that any point on its curve of intersection with the cylinder $x^2+y^2=2ay$ is given by

$$x=2a\sin\theta\cos\theta,\quad y=2a\cos^2\theta,\quad z=k(\sec\theta-\tan\theta).$$

11. P, P′ are (a, b, c), $(-a, -b, -c)$; A, A′ are $(a, b, -c)$, $(-a, b, -c)$; B, B′ are $(-a, b, c)$, $(-a, -b, c)$; and C, C′ are $(a, -b, c)$, $(a, -b, -c)$. Prove that the equation to the surface generated by a conic which passes through P and P′ and intersects the lines AA′, BB′, CC′ is

$$\left(\frac{x}{a}+1\right)\left(\frac{y}{b}-1\right)\left(\frac{z}{c}-\frac{x}{a}\right)\left(\frac{y}{b}-\frac{z}{c}\right)-\left(\frac{x}{a}-\frac{y}{b}\right)^2\left(\frac{z^2}{c^2}-1\right)=0.$$

Shew that this surface contains the lines, AA′, BB′, CC′, PA, PB, PC, P′A′, P′B′, P′C′, PP′. Examine the shape of the surface at the origin. Shew that any point on PP′ is a singular point, and that P and P′ are singular points of the second order, (that is, that the locus of the tangents at P and P′ is a cone of the third degree).

12. If λ, μ are the parameters of the confocals through a point P of an ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, centre O, prove that the points on the wave surface which correspond to the section of the ellipsoid by the diametral plane of OP are given by

$$x^2=\frac{b^2c^2(a^2-\lambda)(a^2-\mu)}{\lambda(a^2-b^2)(a^2-c^2)},\quad y^2=\frac{c^2a^2(b^2-\lambda)(b^2-\mu)}{\lambda(b^2-a^2)(b^2-c^2)},\quad z^2=\frac{a^2b^2(c^2-\lambda)(c^2-\mu)}{\lambda(c^2-a^2)(c^2-b^2)};$$

and the corresponding expressions obtained by interchanging λ and μ.

CHAPTER XIV.

CURVES IN SPACE.

186. The equations to a curve. The equations

$$f_1(x, y, z)=0, \quad f_2(x, y, z)=0$$

together represent the curve of intersection of the surfaces given by $f_1(x, y, z)=0$ and $f_2(x, y, z)=0$. If we eliminate first x, and then y, between the two equations, we obtain equations of the form

$$y=f_3(z), \quad x=f_4(z). \quad \text{.....................(1)}$$

If, now, z be made to depend upon a variable t, z and t being connected by the equation $z=\phi_3(t)$, the equations (1) take the form
$$y=\phi_2(t), \quad x=\phi_1(t).$$

Hence the coordinates of any point on the curve of intersection of two surfaces can be expressed as functions of a single parameter.

Conversely, the locus of a point whose coordinates are given by
$$x=\phi_1(t), \quad y=\phi_2(t), \quad z=\phi_3(t),$$
where t is a parameter, is the curve of intersection of two surfaces. For the elimination of t leads to two equations of the form
$$f_1(x, y)=0, \quad f_2(y, z)=0,$$
which represent two cylinders whose curve of intersection is the locus of the point. (Compare §§ 40, 41, 76, 165.)

187. The tangent. *To find the equations to the tangent at a given point to a given curve.*

Suppose that x, y, z are given as functions of a parameter t. We shall throughout use the symbols x', x'', ...

etc. to denote $\dfrac{dx}{dt}, \dfrac{d^2x}{dt^2}, \ldots$ etc., unless where another meaning is expressly assigned to them. Let the given point, P, be (x, y, z), and let Q, $(x+\delta x, x+\delta y, z+\delta z)$ be a point on the curve adjacent to P. Then, if $x=f(t)$,

$$\begin{aligned} x+\delta x &= f(t+\delta t), \\ &= f(t)+\delta t f'(t)+\frac{\delta t^2}{\lfloor 2}f''(t)+\ldots, \\ &= x+x'\delta t+x''\frac{\delta t^2}{\lfloor 2}+\ldots. \end{aligned}$$

Similarly, $$y+\delta y=y+y'\delta t+y''\frac{\delta t^2}{\lfloor 2}+\ldots,$$

$$z+\delta z=z+z'\delta t+z''\frac{\delta t^2}{\lfloor 2}+\ldots.$$

The equations to PQ are

$$\frac{\xi-x}{x'+x''\dfrac{\delta t}{2}+\ldots}=\frac{\eta-y}{y'+y''\dfrac{\delta t}{2}+\ldots}=\frac{\zeta-z}{z'+z''\dfrac{\delta t}{2}+\ldots}.$$

Now, as Q tends to P, δt tends to zero, and the limiting position of PQ, that is, the tangent at P, is given by

$$\frac{\xi-x}{x'}=\frac{\eta-y}{y'}=\frac{\zeta-z}{z'}.$$

If the equations to the curve are

$$F_1(x, y, z)=0, \quad F_2(x, y, z)=0,$$

we have $$x'\frac{\partial F_1}{\partial x}+y'\frac{\partial F_1}{\partial y}+z'\frac{\partial F_1}{\partial z}=0,$$

$$x'\frac{\partial F_2}{\partial x}+y'\frac{\partial F_2}{\partial y}+z'\frac{\partial F_2}{\partial z}=0;$$

therefore

$$\frac{x'}{\dfrac{\partial F_1}{\partial y}\dfrac{\partial F_2}{\partial z}-\dfrac{\partial F_1}{\partial z}\dfrac{\partial F_2}{\partial y}}=\frac{y'}{\dfrac{\partial F_1}{\partial z}\dfrac{\partial F_2}{\partial x}-\dfrac{\partial F_1}{\partial x}\dfrac{\partial F_2}{\partial z}}=\frac{z'}{\dfrac{\partial F_1}{\partial x}\dfrac{\partial F_2}{\partial y}-\dfrac{\partial F_1}{\partial y}\dfrac{\partial F_2}{\partial x}},$$

whence the direction-ratios of the tangent are found.

Cor. The tangent at a point P to the curve of intersection of two surfaces is the line of intersection of their tangent planes at P.

Ex. 1. Find the equations to the tangent at the point "θ" on the helix
$$x=a\cos\theta,\quad y=a\sin\theta,\quad z=k\theta.$$

Ex. 2. Shew that the tangent at a point of the curve of intersection of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ and the confocal whose parameter is λ is given by
$$\frac{x(\xi-x)}{a^2(b^2-c^2)(a^2-\lambda)}=\frac{y(\eta-y)}{b^2(c^2-a^2)(b^2-\lambda)}=\frac{z(\zeta-z)}{c^2(a^2-b^2)(c^2-\lambda)}.$$

Ex. 3. Shew that the tangent at any point of the curve whose equations, referred to rectangular axes, are
$$x=3t,\quad y=3t^2,\quad z=2t^3$$
makes a constant angle with the line
$$y=z-x=0.$$

188. The direction-cosines of the tangent. If the axes are rectangular, and P, (x, y, z) and Q, $(x+\delta x, y+\delta y, z+\delta z)$ are adjacent points of a given curve, δr, the measure of PQ, is given by
$$\delta r^2=\delta x^2+\delta y^2+\delta z^2.$$

Let the measure of the arc PQ of the curve be δs. Then $\mathrm{Lt}\dfrac{\delta r}{\delta s}=1$, and therefore
$$1=\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2,$$
or
$$s'^2=x'^2+y'^2+z'^2,$$
where x, y, z are functions of t and $x'\equiv\dfrac{dx}{dt}$, etc. Hence the actual direction-cosines of the tangent at P are
$$\frac{x'}{s'},\ \frac{y'}{s'},\ \frac{z'}{s'},\quad\text{or}\quad\frac{dx}{ds},\ \frac{dy}{ds},\ \frac{dz}{ds}.$$

√ **Ex. 1.** For the helix $x=a\cos\theta$, $y=a\sin\theta$, $z=a\theta\tan\alpha$, prove that $\dfrac{ds}{d\theta}=a\sec\alpha$, and that the length of the curve measured from the point where $\theta=0$ is $a\theta\sec\alpha$. (Compare fig. 51.)

Ex. 2. Prove that the length of the curve
$$x=2a(\sin^{-1}t+t\sqrt{1-t^2}),\quad y=2at^2,\quad z=4at,$$
between the points where $t=t_1$ and $t=t_2$, is $4\sqrt{2}a(t_2-t_1)$. Shew also that the curve is a helix drawn on a cylinder whose base is a cycloid and making an angle of 45° with the generators.

189. The normal plane. The locus of the normals to a curve at a point P is the plane through P at right angles to

the tangent at P. If the axes are rectangular the equation to the normal plane is

$$(\xi - x)x' + (\eta - y)y' + (\zeta - z)z' = 0.$$

190. Contact of a curve and surface. If P, P_1, P_2, ... P_n, points of a given curve, lie on a given surface and P_1, P_2, ... P_n tend to P, then in the limit, when P_1, P_2, ... P_n coincide with P, the curve and surface have contact of the n^{th} order at P.

To find the conditions that a curve and surface should have contact of a given order.

Let the equations to the curve and surface be

$$x = \phi_1(t),\ y = \phi_2(t),\ z = \phi_3(t)\ ;\ f(x, y, z) = 0\ ;$$

and let $$\mathrm{F}(t) \equiv f\{\phi_1(t), \phi_2(t), \phi_3(t)\}.$$

Then the roots of the equation $\mathrm{F}(t) = 0$ are the values of t which correspond to the points of intersection of the curve and surface. If the curve and surface have contact of the first order at the point for which $t = t_1$, the equation $\mathrm{F}(t) = 0$ has two roots equal to t_1, and therefore

$$\mathrm{F}(t_1) = 0 \text{ and } \frac{d\mathrm{F}}{dt_1} = 0,$$

and clearly $$\frac{d\mathrm{F}}{dt_1} = \frac{\partial f}{\partial x}\frac{dx}{dt_1} + \frac{\partial f}{\partial y}\frac{dy}{dt_1} + \frac{\partial f}{\partial z}\frac{dz}{dt_1}.$$

If the contact is of the second order, the equation $\mathrm{F}(t) = 0$ has three roots equal to t_1, and therefore

$$\mathrm{F}(t_1) = 0,\quad \frac{d\mathrm{F}}{dt_1} = 0,\quad \frac{d^2\mathrm{F}}{dt_1^2} = 0.$$

And generally, if the contact is of the n^{th} order,

$$\mathrm{F}(t_1) = 0,\quad \frac{d\mathrm{F}}{dt_1} = 0,\ \ldots\ \frac{d^n\mathrm{F}}{dt_1^n} = 0.$$

Ex. 1. Find the plane that has three-point contact at the origin with the curve

$$x = t^4 - 1,\quad y = t^3 - 1,\quad z = t^2 - 1.$$

Ans. $3x - 8y + 6z = 0$.

Ex. 2. Determine a, h, b so that the paraboloid $2z=ax^2+2hxy+by^2$ may have closest possible contact at the origin with the curve

$$x=t^3-2t^2+1, \quad y=t^3-1, \quad z=t^2-2t+1.$$

What is the order of the contact?

Ans. $a/45=h/-3=b/5=1/54$. Fourth.

Ex. 3. Find the inflexional tangents at (x_1, y_1, z_1) on the surface

$$y^2z=4cx.$$

The equations to a line through (x_1, y_1, z_1) may be written

$$x=x_1+lt, \quad y=y_1+mt, \quad z=z_1+nt.$$

The inflexional tangents are the lines which have three-point contact with the surface where $t=0$. For all values of t, we have

$$\frac{dx}{dt}=l, \quad \frac{dy}{dt}=m, \quad \frac{dz}{dt}=n.$$

Hence for three-point contact at (x_1, y_1, z_1), we have

$$\text{(i)} \quad y_1{}^2z_1-4cx_1=0,$$
$$\text{(ii)} \quad -4cl+2y_1z_1m+y_1{}^2n=0,$$
$$\text{(iii)} \quad z_1m^2+2y_1mn=0.$$

Therefore $$\frac{l}{y_1{}^2}=\frac{m}{0}=\frac{n}{4c} \quad \text{or} \quad \frac{l}{3x_1}=\frac{m}{2y_1}=\frac{n}{-z_1}.$$

(Compare § 181, Ex. 2.)

Ex. 4. Find the lines that have four-point contact at (0, 0, 1) with the surface

$$x^4+3xyz+x^2-y^2-z^2+2yz-3xy-2y+2z=1.$$

Ans. The direction-ratios satisfy $lmn=0$, $l^2-m^2-n^2+2mn=0$.

Ex. 5. Prove that if the circle $lx+my+nz=0$, $x^2+y^2+z^2=2cz$ has three-point contact at the origin with the paraboloid

$$ax^2+by^2=2z, \quad c=\frac{l^2+m^2}{bl^2+am^2}.$$

Deduce the result of § 88, Ex. 5.

191. The osculating plane. If P, Q, R are points of a curve, and Q and R tend to P, the limiting position of the plane PQR is the **osculating plane** at the point P.

To find the equation to the osculating plane.

Let the coordinates be functions of a parameter t and P be (x, y, z). The equation to any plane is of the form

$$a\xi+b\eta+c\zeta+d=0.$$

If this plane and the curve have contact of the second order at (x, y, z), we have

$$\begin{aligned} ax+by+cz+d&=0, \\ ax'+by'+cz' \quad &=0, \\ ax''+by''+cz'' \quad &=0. \end{aligned}$$

Therefore, eliminating a, b, c, d, we obtain the equation to the osculating plane,

$$\begin{vmatrix} \xi-x, & \eta-y, & \zeta-z \\ x', & y', & z' \\ x'', & y'', & z'' \end{vmatrix}=0.$$

Ex. 1. Find the osculating plane at the point "θ" on the helix $x=a\cos\theta$, $y=a\sin\theta$, $z=k\theta$.

Ans. $k(x\sin\theta-y\cos\theta-a\theta)+az=0.$

Ex. 2. For the curve $x=3t$, $y=3t^2$, $z=2t^3$, shew that any plane meets it in three points and deduce the equation to the osculating plane at $t=t_1$. *Ans.* $2t_1^2x-2t_1y+z=2t_1^3$.

Ex. 3. Prove that there are three points on the cubic $x=at^3+b$, $y=3ct^2+3dt$, $z=3et+f$, such that the osculating planes pass through the origin, and that the points lie in the plane $3cex+afy=0$.

Ex. 4. P and Q are points of a curve and PT is the tangent at P. Prove that the limiting position of the plane PQT as Q tends to P is the osculating plane at P.

Ex. 5. Normals are drawn from the point (α, β, γ) to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$. Find the equation to the osculating plane at (α, β, γ) of the cubic curve through the feet of the normals.

Ans. $\dfrac{a^4x}{(c^2-a^2)(a^2-b^2)\alpha}+\dfrac{b^4y}{(a^2-b^2)(b^2-c^2)\beta}+\dfrac{c^4z}{(b^2-c^2)(c^2-a^2)\gamma}+1=0.$

Ex. 6. Shew that the condition that four consecutive points of a curve should be coplanar is

$$\begin{vmatrix} x', & y', & z' \\ x'', & y'', & z'' \\ x''', & y''', & z''' \end{vmatrix}=0.$$

Ex. 7. Prove that the equations

$$x=a_1t^2+2b_1t+c_1,\quad y=a_2t^2+2b_2t+c_2,\quad z=a_3t^2+2b_3t+c_3$$

determine a parabola, and find the equation to the plane in which it lies.

Ex. 8. Shew that the curve for which

$$e^{ax}=\frac{b-t}{c-t},\quad e^{by}=\frac{c-t}{a-t},\quad e^{cz}=\frac{a-t}{b-t}$$

is a plane curve which lies in the plane $ax+by+cz=0$.

192. *To find the osculating plane at a point of the curve of intersection of the surfaces $f(\xi, \eta, \zeta)=0$, $\phi(\xi, \eta, \zeta)=0$.*

The equations to the tangent at (x, y, z) are

$$(\xi-x)f_x+(\eta-y)f_y+(\zeta-z)f_z=0,$$

$$(\xi-x)\phi_x+(\eta-y)\phi_y+(\zeta-z)\phi_z=0,$$

and therefore the equation to the osculating plane is of the form

$$\lambda\{(\xi-x)f_x+(\eta-y)f_y+(\zeta-z)f_z\}$$
$$=\mu\{(\xi-x)\phi_x+(\eta-y)\phi_y+(\zeta-z)\phi_z\}.$$

That this plane should have contact of the second order with the curve, we must have

$$\lambda\{x'f_x+y'f_y+z'f_z\}=\mu\{x'\phi_x+y'\phi_y+z'\phi_z\} \quad \ldots\ldots(1)$$

and $$\lambda\{x''f_x+y''f_y+z''f_z\}=\mu\{x''\phi_x+y''\phi_y+z''\phi_z\}. \quad \ldots(2)$$

But $x'f_x+y'f_y+z'f_z=0$ and $x'\phi_x+y'\phi_y+z'\phi_z=0$, ...(3) and therefore equation (1) is an identity. This is to be expected, since any plane through the tangent to a curve has contact of the first order with the curve. Differentiating the equations (3), we obtain

$$x'^2f_{xx}+y'^2f_{yy}+z'^2f_{zz}+2y'z'f_{yz}+2z'x'f_{zx}+2x'y'f_{xy}$$
$$=-(x''f_x+y''f_y+z''f_z),$$
$$x'^2\phi_{xx}+y'^2\phi_{yy}+z'^2\phi_{zz}+2y'z'\phi_{yz}+2z'x'\phi_{zx}+2x'y'\phi_{xy}$$
$$=-(x''\phi_x+y''\phi_y+z''\phi_z),$$

whence by (2) the equation to the osculating plane is

$$\frac{(\xi-x)f_x+(\eta-y)f_y+(\zeta-z)f_z}{x'^2f_{xx}\ldots+2y'z'f_{yz}+\ldots}=\frac{(\xi-x)\phi_x+(\eta-y)\phi_y+(\zeta-z)\phi_z}{x'^2\phi_{xx}\ldots+2y'z'\phi_{yz}+\ldots}.$$

Ex. 1. Prove that the osculating plane at (x_1, y_1, z_1) on the curve of intersection of the cylinders $x^2+z^2=a^2$, $y^2+z^2=b^2$ is given by

$$\frac{xx_1^3-zz_1^3-a^4}{a^2}=\frac{yy_1^3-zz_1^3-b^4}{b^2}.$$

Ex. 2. Find the osculating plane at a point of the curve of intersection of the conicoids

$$f\equiv ax^2+by^2+cz^2-1=0, \quad \phi\equiv\alpha x^2+\beta y^2+\gamma z^2-1=0.$$

We have $$axx'+byy'+czz'=0,$$
$$\alpha xx'+\beta yy'+\gamma zz'=0,$$
whence $$\frac{xx'}{A}=\frac{yy'}{B}=\frac{zz'}{C},$$
where $A\equiv b\gamma-c\beta$, $B\equiv c\alpha-a\gamma$, $C\equiv a\beta-b\alpha$.

Again, $f_{xx}=2a$, $f_{yy}=2b$, $f_{zz}=2c$; $f_{yz}=f_{zx}=f_{xy}=0$; $\phi_{xx}=2\alpha$, etc.

Therefore the required equation is

$$\frac{(\xi-x)ax+(\eta-y)by+(\zeta-z)cz}{\dfrac{aA^2}{x^2}+\dfrac{bB^2}{y^2}+\dfrac{cC^2}{z^2}}=\frac{(\xi-x)\alpha x+(\eta-y)\beta y+(\zeta-z)\gamma z}{\dfrac{\alpha A^2}{x^2}+\dfrac{\beta B^2}{y^2}+\dfrac{\gamma C^2}{z^2}},$$

which reduces to $\Sigma(\xi-x)x^3BC(Bz^2-Cy^2)=0$.

The equation may be further transformed.

We have $\alpha f - a\phi \equiv Bz^2 - Cy^2 - (\alpha - a) = 0$, etc.

Hence the equation may be written

$$\Sigma \xi x^3 BC(\alpha - a) = \Sigma x^4 BC(Bz^2 - Cy^2),$$
$$= -(Bz^2 - Cy^2)(Cx^2 - Az^2)(Ay^2 - Bx^2),$$
$$= -(\alpha - a)(\beta - b)(\gamma - c),$$

or
$$\frac{BCx^3\xi}{(\beta - b)(\gamma - c)} + \frac{CAy^3\eta}{(\gamma - c)(\alpha - a)} + \frac{ABz^3\zeta}{(\alpha - a)(\beta - b)} + 1 = 0.$$

Ex. 3. Shew that at (x', y', z'), a point of intersection of the three confocals,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad \frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1, \quad \frac{x^2}{a^2+\mu} + \frac{y^2}{b^2+\mu} + \frac{z^2}{c^2+\mu} = 1,$$

the osculating plane of the curve of intersection of the first two is given by

$$\frac{xx'(a^2+\mu)}{a^2(a^2+\lambda)} + \frac{yy'(b^2+\mu)}{b^2(b^2+\lambda)} + \frac{zz'(c^2+\mu)}{c^2(c^2+\lambda)} = 1.$$

Ex. 4. Prove that the points of the curve of intersection of the sphere and conicoid

$$rx^2 + ry^2 + rz^2 = 1, \quad ax^2 + by^2 + cz^2 = 1,$$

at which the osculating planes pass through the origin, lie on the cone

$$\frac{a-r}{b-c}x^4 + \frac{b-r}{c-a}y^4 + \frac{c-r}{a-b}z^4 = 0.$$

193. The principal normal and binormal. There is an infinite number of normals to a curve at a given point, A,

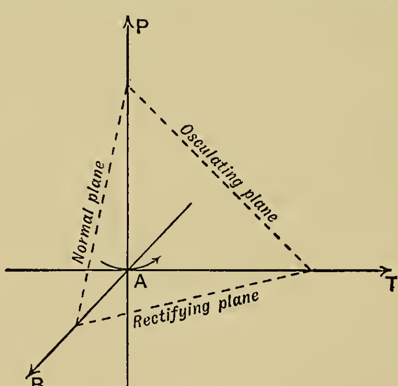

FIG. 54.

on it, and their locus is the normal plane at A. Two of the normals are of special importance, that which lies in the osculating plane at A and is called the **principal normal**, and

that which is perpendicular to the osculating plane and is called the **binormal**. In fig. 54 AT is the tangent, AP the principal normal, AB the binormal; the plane ATP is the osculating plane, and the plane ABP is the normal plane. The plane ABT is called the **rectifying plane**.

We shall choose as the positive direction of the tangent at A the direction in which the arc increases, and as the positive direction of the principal normal, that towards which the concavity of the curve is turned. We shall then choose the positive direction of the binormal so that the positive directions of the tangent, principal normal and binormal can be brought by rotation into coincidence with the positive directions of the x-, y-, and z-axes respectively.

Let us throughout denote the direction-cosines of the

tangent by	$l_1,\ m_1,\ n_1$;
principal normal by	$l_2,\ m_2,\ n_2$;
binormal by	$l_3,\ m_3,\ n_3$.

Then, (§ 188),

$$l_1=+\frac{dx}{ds},\quad m_1=+\frac{dy}{ds},\quad n_1=+\frac{dz}{ds}.$$

Again, (§ 191),

$$\frac{l_3}{y'z''-z'y''}=\frac{m_3}{z'x''-x'z''}=\frac{n_3}{x'y''-y'x''},\ \text{where } x'\equiv\frac{dx}{dt},\ \text{etc.},$$

$$=\frac{\pm 1}{\sqrt{(x'^2+y'^2+z'^2)(x''^2+y''^2+z''^2)-(x'x''+y'y''+z'z'')^2}},$$

(by Lagrange's identity).

But, (§ 188), $\qquad x'^2+y'^2+z'^2=s'^2,$

and therefore $\qquad x'x''+y'y''+z'z''=s's''.$

$$\therefore\ \frac{l_3}{y'z''-z'y''}=\frac{m_3}{z'x''-x'z''}=\frac{n_3}{x'y''-y'x''}$$

$$=\frac{\pm 1}{s'\sqrt{x''^2+y''^2+z''^2-s''^2}}.$$

We have also $\qquad l_2=m_3n_1-m_1n_3.$

Therefore

$$l_2 = \pm\frac{z'(z'x''-x'z'')-y'(x'y''-y'x'')}{s'^2\sqrt{x''^2+y''^2+z''^2-s''^2}},$$

$$= \pm\frac{x''(x'^2+y'^2+z'^2)-x'(x'x''+y'y''+z'z'')}{s'^2\sqrt{x''^2+y''^2+z''^2-s''^2}},$$

$$= \pm\frac{x''s'-x's''}{s'\sqrt{x''^2+y''^2+z''^2-s''^2}};$$

and similarly,

$$m_2 = \pm\frac{y''s'-y's''}{s'\sqrt{x''^2+y''^2+z''^2-s''^2}},\quad n_2 = \pm\frac{z''s'-z's''}{s'\sqrt{x''^2+y''^2+z''^2-s''^2}}.$$

Ex. 1. Prove that the parallels through the origin to the binormals of the helix $x=a\cos\theta,\ y=a\sin\theta,\ z=k\theta$ lie upon the right cone $a^2(x^2+y^2)=k^2z^2$.

Ex. 2. Prove that the principal normal to the helix is the normal to the cylinder.

194. Curvature. If A_1 and A_2 are points of a given curve so that the arc A_1A_2 is positive and of length δs and the angle between the tangents at A_1 and A_2 is $\delta\psi$, the ratio $\frac{\delta\psi}{\delta s}$ gives the average rate of change in the direction of the tangent over the arc A_1A_2. The rate of change at A_1 is measured by the Lt $\frac{\delta\psi}{\delta s}$, that is by $\frac{d\psi}{ds}$, and is called the **curvature** of the curve at A. It is denoted by $1/\rho$, and ρ is called the **radius of curvature.**

195. Torsion. The direction of the osculating plane at a point of any curve which is not plane changes as the point describes the curve. If $\delta\tau$ is the angle between the binormals at A_1 and A_2, the ratio $\frac{\delta\tau}{\delta s}$ gives the average rate of change of direction of the osculating plane over the arc A_1A_2. The rate of change at A_1 is measured by the Lt $\frac{\delta\tau}{\delta s}$, that is by $\frac{d\tau}{ds}$, and is called the **torsion** at A_1. It is denoted by $1/\sigma$, and σ is called the **radius of torsion.**

196. The spherical indicatrices. The formulae for curvature and torsion are readily deduced by means of spherical indicatrices, which are constructed as follows. From the origin, O, draw in the positive directions of the tangents to the curve, radii of the sphere of unit radius whose centre is O. The extremities of these radii form a curve on the sphere which is the spherical indicatrix of the tangents. Similarly, by drawing radii in the positive directions of the binormals, we construct the spherical indicatrix of the binormals.

197. Frenet's formulae. In figs. 55, 56, let A_1, A_2, A_3, ... be adjacent points of a given curve, and let Ot_1, Ot_2, Ot_3, ... be drawn in the same directions as the tangents

$$A_1T_1,\ A_2T_2,\ A_3T_3,\ \dots,$$

and Ob_1, Ob_2, Ob_3, ..., in the same directions as the binormals

$$A_1B_1,\ A_2B_2,\ A_3B_3,\ \dots.$$

Then $t_1, t_2, t_3, \dots, b_1, b_2, b_3, \dots$ are adjacent points on the indicatrices of the tangents and binormals.

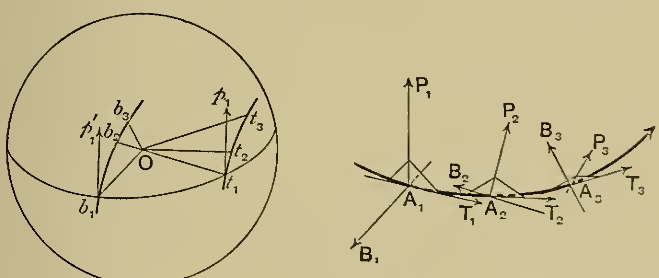

FIG. 55.

Since Ot_1 and Ot_2 are parallel to adjacent tangents to the curve, the limiting position of the plane t_1Ot_2 is parallel to the osculating plane of the curve at A_1. Hence the tangent at t_1 to the indicatrix $t_1t_2t_3\dots$, being the limiting position of t_1t_2, is at right angles to the binormal at A_1. And since it is a tangent to the sphere, it is at right angles to the radius Ot_1, and is therefore at right angles to the tangent A_1T_1. Therefore the tangent at t_1 to the indicatrix

$t_1t_2t_3 \ldots$ is parallel to A_1P_1, the principal normal at A_1. Let us take as the positive direction of the tangent to the indicatrix, the positive direction of the principal normal.

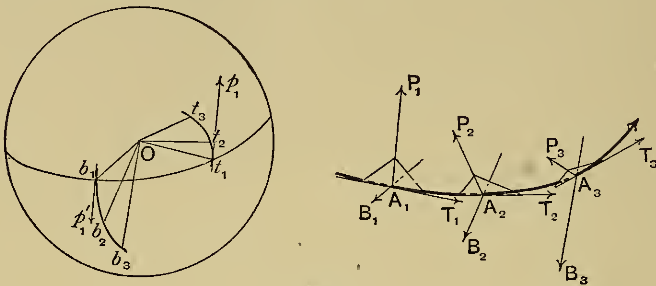

FIG. 56.

Since the sphere is of unit radius, the measures of the arc t_1t_2 of the great circle in which the plane t_1Ot_2 cuts the sphere, and of the angle t_1Ot_2 are equal, and hence the measure of the arc is $\delta\psi$. Let $\delta\alpha$ measure the arc t_1t_2 of the indicatrix. Then $\text{Lt}\dfrac{\delta\alpha}{\delta\psi} = \pm 1$. If we take the arcs of the indicatrix and great circle through t_1, t_2 in the same sense, the limit is $+1$, and since we have fixed the positive direction of the tangent to the indicatrix at t_1, we thus fix the sign of $\delta\psi$. Hence we have, in magnitude and sign,

$$\rho = \text{Lt}\frac{\delta s}{\delta\psi} = \text{Lt}\frac{\delta s}{\delta\alpha} = \frac{ds}{d\alpha}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Again, t_1 is the point (l_1, m_1, n_1), and therefore, by § 188, the direction-cosines of the positive tangent at t_1 to the indicatrix are

$$\frac{dl_1}{d\alpha}, \quad \frac{dm_1}{d\alpha}, \quad \frac{dn_1}{d\alpha}.$$

Whence $$l_2 = \frac{dl_1}{d\alpha}, \quad m_2 = \frac{dm_1}{d\alpha}, \quad n_2 = \frac{dn_1}{d\alpha},$$

or, by (1), $$\frac{l_2}{\rho} = \frac{dl_1}{ds}, \quad \frac{m_2}{\rho} = \frac{dm_1}{ds}, \quad \frac{n_2}{\rho} = \frac{dn_1}{ds}. \quad \ldots\ldots\ldots\ldots(\text{A})$$

Further, if we consider Ot_1, Ot_2, Ot_3 to be generators of a cone of the second degree, the planes t_1Ot_2, t_2Ot_3 are ultimately tangent planes, and therefore their normals Ob_1, Ob_2 are generators of the reciprocal cone. Hence the limiting

position of the plane $b_1\mathrm{O}b_2$, *i.e.* the tangent plane to the reciprocal cone, is at right angles to $\mathrm{O}t_1$, and the limiting position of b_1b_2, *i.e.* the tangent at b_1 to the indicatrix $b_1b_2b_3\dots$ is at right angles to $\mathrm{O}t_1$. Besides, the tangent at b_1 is a tangent to the sphere, and is therefore at right angles to $\mathrm{O}b_1$. Therefore the tangent at b_1 to the indicatrix $b_1b_2b_3\dots$ is parallel to $\mathrm{A}_1\mathrm{P}_1$, the principal normal at A_1. Suppose that its positive direction is that of the principal normal.

If the measure, with the proper sign, of the arc b_1b_2 of the indicatrix is $\delta\beta$, then the measure of the arc b_1b_2 of the great circle in the plane $b_1\mathrm{O}b_2$ is $\delta\tau$, and $\mathrm{Lt}\dfrac{\delta\beta}{\delta\tau}=\pm 1$. If we take the arcs in the same sense so that the limit is $+1$, since we have assigned a positive direction to the tangent at b_1 to the indicatrix, we fix the sign of $\delta\tau$. Hence we have, in magnitude and sign,

$$\sigma=\mathrm{Lt}\frac{\delta s}{\delta\tau}=\mathrm{Lt}\frac{\delta s}{\delta\beta}=\frac{ds}{d\beta}. \quad\dots\dots\dots\dots\dots\dots(2)$$

Again, the coordinates of b_1 are l_3, m_3, n_3, and hence the direction-cosines of the positive tangent at b_1 to the indicatrix are

$$\frac{dl_3}{d\beta},\quad \frac{dm_3}{d\beta},\quad \frac{dn_3}{d\beta}.$$

Therefore $\quad l_2=\dfrac{dl_3}{d\beta},\quad m_2=\dfrac{dm_3}{d\beta},\quad n_2=\dfrac{dn_3}{d\beta}.$

Or, by (2), $\quad \dfrac{l_2}{\sigma}=\dfrac{dl_3}{ds},\quad \dfrac{m_2}{\sigma}=\dfrac{dm_3}{ds},\quad \dfrac{n_2}{\sigma}=\dfrac{dn_3}{ds}. \quad\dots\dots\dots\dots\dots(\mathrm{B})$

We have also, $\quad l_1^2+l_2^2+l_3^2=1.$

Hence $\quad l_1\dfrac{dl_1}{ds}+l_2\dfrac{dl_2}{ds}+l_3\dfrac{dl_3}{ds}=0.$

Therefore, by (A) and (B),

$$\frac{dl_2}{ds}=-\frac{l_1}{\rho}-\frac{l_3}{\sigma}.$$

Similarly, $\dfrac{dm_2}{ds}=-\dfrac{m_1}{\rho}-\dfrac{m_3}{\sigma},\quad \dfrac{dn_2}{ds}=-\dfrac{n_1}{\rho}-\dfrac{n_3}{\sigma}. \quad\dots\dots\dots\dots(\mathrm{C})$

The results (A), (B), (C) are exceedingly important. They are known as **Frenet's Formulae**.

198. The signs of the curvature and torsion. We have agreed that the positive direction of the tangent at t_1 to the indicatrix $t_1t_2t_3\ldots$ is that of the principal normal. But if the positive direction of the principal normal is that towards which the concavity of the curve is turned, the direction at t_1 of the arc t_1t_2 is that of the positive direction of the principal normal, (see figs. 55, 56). Therefore, $\delta\alpha$, $\delta\psi$ and ρ are always positive.

We also agreed that the positive direction of the tangent at b_1 to the indicatrix $b_1b_2b_3\ldots$ was the positive direction of the principal normal. The direction at b_1 of the arc b_1b_2 is the positive direction of the principal normal for a curve such as that in fig. 55, but is the opposite direction for a curve such as that in fig. 56. For the curve in fig. 55, the apparent rotation of the principal normal and binormal as the arc increases is that of a left-handed screw, and such curves are therefore called **sinistrorsum.** For such a curve $\delta\beta$, $\delta\tau$ and σ are positive. For the curve in fig. 56, the apparent rotation is that of a right-handed screw, and such curves are said to be **dextrorsum.** For this class of curve $\delta\beta$, $\delta\tau$ and σ are negative.

199. To find the radius of curvature.

From (A), § 197, by squaring and adding we obtain

$$\frac{1}{\rho^2}=\left(\frac{dl_1}{ds}\right)^2+\left(\frac{dm_1}{ds}\right)^2+\left(\frac{dn_1}{ds}\right)^2. \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

But $\qquad l_1=\dfrac{x'}{s'}$, therefore $\dfrac{dl_1}{ds}=\dfrac{x''s'-s''x'}{s'^3}$.

Hence
$$\frac{1}{\rho^2}=\Sigma\frac{(x''s'-s''x')^2}{s'^6},$$
$$=\frac{1}{s'^6}\{s'^2\Sigma x''^2-2s's''\Sigma x'x''+s''^2\Sigma x'^2\}.$$

Therefore, since $\Sigma x'^2=s'^2$, and $\Sigma x'x''=s's''$,

$$\frac{1}{\rho^2}=\frac{x''^2+y''^2+z''^2-s''^2}{s'^4}.$$

Cor. If the coordinates are functions of s, the length of the arc measured from a fixed point, so that $t \equiv s$, then $s' = 1$, $s'' = 0$, and

$$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2.$$

The student should note the analogy between these formulae and those for the radius of curvature of a plane curve.

Ex. Deduce equation (1) from the result of Ex. 10, § 23.

200. *To find the direction-cosines of the principal normal and binormal.*

From equations (A), § 197,

$$l_2 = \rho\left(\frac{dl_1}{ds}\right) = \rho\frac{x''s' - s''x'}{s'^3}.$$

Similarly, $\quad m_2 = \rho\dfrac{y''s' - s''y'}{s'^3}, \quad n_2 = \rho\dfrac{z''s' - s''z'}{s'^3}.$

Again, since $\quad m_1 = \dfrac{y'}{s'}, \quad n_1 = \dfrac{z'}{s'}, \quad$ and $\quad l_3 = m_1 n_2 - m_2 n_1,$

$$l_3 = \rho\frac{y'z'' - z'y''}{s'^3}.$$

Similarly, $\quad m_3 = \rho\dfrac{z'x'' - x'z''}{s'^3}, \quad n_3 = \rho\dfrac{x'y'' - y'x''}{s'^3}.$

Compare § 193.

Cor. If $t \equiv s$, we have.

$$l_2 = \rho\frac{d^2x}{ds^2}, \quad m_2 = \rho\frac{d^2y}{ds^2}, \quad n_2 = \rho\frac{d^2z}{ds^2},$$

$$l_3 = \rho\left(\frac{dy}{ds}\cdot\frac{d^2z}{ds^2} - \frac{dz}{ds}\cdot\frac{d^2y}{ds^2}\right), \quad \text{etc.}$$

201. To find the radius of torsion. From Frenet's formulae, (B), we have $\dfrac{dl_3}{ds} = \dfrac{l_2}{\sigma}$, and from § 200,

$$l_3 = \frac{\rho}{s'^3}(y'z'' - z'y''), \quad \text{or} \quad l_3 s'^3 = \rho(y'z'' - z'y'').$$

Differentiating with respect to t, we obtain

$$\frac{l_2 s'^4}{\sigma} + 3l_3 s'^2 s'' = \rho(y'z''' - z'y''') + \rho'(y'z'' - z'y''),$$

$$= \rho(y'z''' - z'y''') + \frac{l_3\rho' s'^3}{\rho}. \quad \text{..................(1)}$$

Similarly, $\dfrac{m_2 s'^4}{\sigma} + 3m_3 s'^2 s'' = \rho(z'x''' - x'z''') + \dfrac{m_3 \rho' s'^3}{\rho}$, ...(2)

and $\dfrac{n_2 s'^4}{\sigma} + 3n_3 s'^2 s'' = \rho(x'y''' - y'x''') + \dfrac{n_3 \rho' s'^3}{\rho}$. ...(3)

Multiply (1), (2), (3) by l_2, m_2, n_2 respectively, and add, and we have

$$\frac{s'^4}{\sigma} = \rho\{l_2(y'z''' - z'y''') + m_2(z'x''' - x'z''') + n_2(x'y''' - y'x''')\},$$

which, on substituting $\rho\dfrac{x''s' - s''x'}{s'^3}$ for l_2, etc., becomes

$$\frac{1}{\sigma} = -\frac{\rho^2}{s'^6}\begin{vmatrix} x', & y', & z' \\ x'', & y'', & z'' \\ x''', & y''', & z''' \end{vmatrix}.$$

Ex. 1. Find the radii of curvature and torsion of the helix

$$x = a\cos\theta, \quad y = a\sin\theta, \quad z = a\theta\tan\alpha.$$

We have $x' = -a\sin\theta, \quad y' = a\cos\theta, \quad z' = a\tan\alpha.$

Therefore $s'^2 = x'^2 + y'^2 + z'^2 = a^2\sec^2\alpha.$

Hence $x'' = -a\cos\theta, \quad y'' = -a\sin\theta, \quad z'' = s'' = 0,$

and $x''' = a\sin\theta, \quad y''' = -a\cos\theta, \quad z''' = 0.$

Therefore $\dfrac{1}{\rho^2} = \dfrac{x''^2 + y''^2 + z''^2 - s''^2}{s'^4} = \dfrac{1}{a^2\sec^4\alpha},$

and $\dfrac{1}{\rho^2\sigma} = -\dfrac{1}{s'^6}\begin{vmatrix} -a\sin\theta, & a\cos\theta, & a\tan\alpha \\ -a\cos\theta, & -a\sin\theta, & 0 \\ a\sin\theta, & -a\cos\theta, & 0 \end{vmatrix} = \dfrac{-\tan\alpha}{a^3\sec^6\alpha};$

whence $\sigma = -a/\sin\alpha\cos\alpha.$

Ex. 2. For the curve $x = 3t$, $y = 3t^2$, $z = 2t^3$, prove that

$$\rho = -\sigma = \tfrac{3}{2}(1 + 2t^2)^2.$$

Ex. 3. For the curve $x = 2a(\sin^{-1}\lambda + \lambda\sqrt{1-\lambda^2})$, $y = 2a\lambda^2$, $z = 4a\lambda$, prove that $\rho = -\sigma = 8a\sqrt{1-\lambda^2}$.

Ex. 4. For a point of the curve of intersection of the surfaces

$$x^2 - y^2 = c^2, \quad y = x\tanh\frac{z}{c}, \quad \rho = -\sigma = \frac{2x^2}{c}.$$

($x = c\cosh t$, $y = c\sinh t$, $z = ct$.)

Ex. 5. For the curve $x = \sqrt{6}a\lambda^3$, $y = a(1 + 3\lambda^2)$, $z = \sqrt{6}a\lambda$, prove that $\sigma = y^2/a$.

Ex. 6. Find the radii of curvature and torsion at a point of the curve $x^2 + y^2 = a^2$, $x^2 - y^2 = az$.

Ans. $\rho^2 = \dfrac{(5a^2 - 4z^2)^3}{a^2(5a^2 + 12z^2)}, \quad \sigma = \dfrac{5a^2 + 12z^2}{6\sqrt{a^2 - z^2}}.$

($x = a\cos\theta$, $y = a\sin\theta$, $z = a\cos 2\theta$.)

202. *If the tangent to a curve makes a constant angle* α *with a fixed line* $\sigma = \pm\rho \tan\alpha$.

Take the fixed line as z-axis. Then

$$n_1 = \cos\alpha, \text{ and } \frac{n_2}{\rho} = \frac{dn_1}{ds} = 0, \text{ (§ 197, (A))}.$$

Therefore $\quad n_2 = 0, \text{ and } n_3 = \pm\sin\alpha.$

Again, $$\frac{n_1}{\rho} + \frac{n_3}{\sigma} = -\frac{dn_2}{ds} = 0, \text{ (§ 197, (C))}.$$

Therefore $$\sigma = \pm\rho\tan\alpha.$$

Ex. 1. For the curves in Exs. 2, 3, 4, § 201, shew that the tangent makes an angle of 45° with a fixed line, and hence that $\rho = \pm\sigma$.

Ex. 2. If a curve is drawn on any cylinder and makes a constant angle α with the generators, $\rho = \rho_0 \operatorname{cosec}^2\alpha$, where $1/\rho$ and $1/\rho_0$ are the curvatures at any point P of the curve and the normal section of the cylinder through P.

Take the z-axis parallel to the generators of the cylinder. Then if δs, δs_1 are infinitesimal arcs of the curve and normal section, $\frac{ds_1}{ds} = \sin\alpha$, $\frac{dz}{ds} = \cos\alpha$ and $\frac{d^2z}{ds^2} = 0$. If P is (x, y, z),

$$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 \text{ and } \frac{1}{\rho_0^2} = \left(\frac{d^2x}{ds_1^2}\right)^2 + \left(\frac{d^2y}{ds_1^2}\right)^2.$$

Whence the result immediately follows.

Ex. 3. Apply Ex. 2 to shew that the curvature of the helix

$$x = a\cos\theta, \quad y = a\sin\theta, \quad z = a\theta\tan\alpha,$$

is $\frac{\cos^2\alpha}{a}$, then deduce that the torsion is $\pm\frac{\sin\alpha\cos\alpha}{a}$.

Ex. 4. If ρ/σ is constant the curve is a helix.

Since $$\frac{l_2}{\rho} = \frac{dl_1}{ds} \text{ and } \frac{l_2}{\sigma} = \frac{dl_3}{ds}, \quad dl_1 = k\,dl_3.$$

Therefore $l_1 = kl_3 + k_1$, where k_1 is an arbitrary constant.

Similarly, $$m_1 = km_3 + k_2, \quad n_1 = kn_3 + k_3.$$

Multiplying by l_1, m_1, n_1, and adding, we obtain $k_1l_1 + k_2m_1 + k_3n_1 = 1$. Hence, since $k_1^2 + k_2^2 + k_3^2 = 1 + k^2$, and therefore k_1, k_2, k_3, cannot all be zero, the tangent to the curve makes a constant angle with the fixed line

$$\frac{x}{k_1} = \frac{y}{k_2} = \frac{z}{k_3}.$$

Parallels drawn through points of the curve to this line generate a cylinder on which the curve lies, hence the curve is a helix.

Ex. 5. If ρ and σ are constant, the curve is a right circular helix.

Ex. 6. A curve is drawn on a parabolic cylinder so as to cut all the generators at the same angle. Find expressions for the curvature and torsion.

Ans. If the cylinder is $x = at^2$, $y = 2at$, and the angle is α,

$$\rho = 2a(1+t^2)^{\frac{3}{2}}/\sin^2\alpha \text{ and } \sigma = 2a(1+t^2)^{\frac{3}{2}}/\sin\alpha\cos\alpha.$$

203. The circle of curvature. If P, Q, R are points of a curve, the limiting position of the circle PQR as Q and R tend to P is the **osculating circle** at P.

From the definitions of the osculating plane and the curvature at P, it follows immediately that the osculating circle lies in the osculating plane at P, and that its radius is the radius of curvature at P. It also follows that the centre of the circle, or the **centre of curvature**, lies on the principal normal, and therefore its coordinates are

$$x+l_2\rho, \quad y+m_2\rho, \quad z+n_2\rho.$$

We can easily deduce the radius and the coordinates of the centre by means of Frenet's formulae. If (α, β, γ) is the centre and r the radius of the circle of curvature, the equations

$$l_3(\xi-x)+m_3(\eta-y)+n_3(\zeta-z)=0, \quad \text{...........(1)}$$
$$(\xi-\alpha)^2+(\eta-\beta)^2+(\zeta-\gamma)^2=r^2 \quad \text{...........(2)}$$

may be taken to represent it. Since the sphere (2) has three-point contact with the curve at (x, y, z), differentiating twice with respect to s and applying Frenet's formulae, we have

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=r^2, \quad \text{...........(3)}$$
$$l_1(x-\alpha)+m_1(y-\beta)+n_1(z-\gamma)=0, \quad \text{...........(4)}$$
$$l_2(x-\alpha)+m_2(y-\beta)+n_2(z-\gamma)=-\rho. \quad \text{........(5)}$$

And since the centre (α, β, γ) lies in the osculating plane, (1), $\quad l_3(x-\alpha)+m_3(y-\beta)+n_3(z-\gamma)=0. \quad$ ....(6)

Square and add (4), (5), (6), and

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=\rho^2.$$

Therefore, by (3), $\qquad r=\rho.$

Multiply (4), (5), (6) by l_1, l_2, l_3 respectively, and add, and

$$x-\alpha=-l_2\rho.$$

Similarly, $\quad y-\beta=-m_2\rho, \quad z-\gamma=-n_2\rho.$

Therefore $\alpha=x+l_2\rho, \quad \beta=y+m_2\rho, \quad \gamma=z+n_2\rho.$

204. The osculating sphere. If P, Q, R, S are points of a curve, the limiting position of the sphere PQRS as Q, R and S tend to P is the **osculating sphere** at P.

To find the centre and radius of the osculating sphere.

Assume that the equation is

$$(\xi-\alpha)^2+(\eta-\beta)^2+(\zeta-\gamma)^2=R^2.$$

Then, for four-point contact at (x, y, z), we have on differentiating three times with respect to s,

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=R^2, \quad \text{.........(1)}$$

$$l_1(x-\alpha)+m_1(y-\beta)+n_1(z-\gamma)=0, \quad \text{.........(2)}$$

$$l_2(x-\alpha)+m_2(y-\beta)+n_2(z-\gamma)=-\rho, \quad \text{.........(3)}$$

$$\left(\frac{l_3}{\sigma}+\frac{l_1}{\rho}\right)(x-\alpha)+\left(\frac{m_3}{\sigma}+\frac{m_1}{\rho}\right)(y-\beta)+\left(\frac{n_3}{\sigma}+\frac{n_1}{\rho}\right)(z-\gamma)=\rho',$$

or, by (2), $l_3(x-\alpha)+m_3(y-\beta)+n_3(z-\gamma)=\sigma\rho'$, ...(4)

where $$\rho' \equiv \frac{d\rho}{ds}.$$

Whence, as in § 203, we deduce

$$R^2=\rho^2+\rho'^2\sigma^2,$$

or $$=\rho^2+\left(\frac{d\rho}{ds}\right)^2\left(\frac{ds}{d\tau}\right)^2=\rho^2+\left(\frac{d\rho}{d\tau}\right)^2,$$

and

$$\alpha=x+l_2\rho-l_3\sigma\rho', \quad \beta=y+m_2\rho-m_3\sigma\rho', \quad \gamma=z+n_2\rho-n_3\sigma\rho'.$$

These shew that the centre of the osculating sphere, or **centre of spherical curvature**, lies on a line drawn through the centre of circular curvature parallel to the binormal, and is distant $-\sigma\rho'$ from the centre of circular curvature.

Cor. If a curve is drawn on a sphere of radius a, $R=a$, and therefore $\sigma^2=\dfrac{a^2-\rho^2}{\rho'^2}$. Hence, if ρ is known, σ can be deduced. Further, if we differentiate $a^2=\rho^2+\left(\dfrac{d\rho}{d\tau}\right)^2$, we eliminate the constant a and obtain a differential equation satisfied by all spherical curves

$$\rho+\frac{d^2\rho}{d\tau^2}=0.$$

Ex. 1. A curve is drawn on a sphere of radius a so as to make a constant angle α with the plane of the equator. Shew that at the point whose north-polar distance is θ, $\rho=a(1-\sec^2\alpha\cos^2\theta)^{\frac{1}{2}}$.

We have, by § 202, $\sigma=\rho\cot\alpha.$

Also $$\sigma^2=\frac{a^2-\rho^2}{\left(\frac{d\rho}{ds}\right)^2}, \text{ and } \frac{dz}{ds}=\sin\alpha.$$

But $$z=a\cos\theta, \text{ therefore } \frac{dz}{ds}=-a\sin\theta\frac{d\theta}{ds}.$$

Whence $$\frac{\rho\, d\rho}{\sqrt{a^2-\rho^2}}=\frac{-a\sin\theta\, d\theta}{\cos\alpha}.$$

Integrating, we obtain

$$\sqrt{a^2-\rho^2}=-a\cos\theta\sec\alpha+b,$$

where b is an arbitrary constant.

If $\rho=a$ when $\theta=\pi/2$, $b=0$, and then

$$\rho=a(1-\cos^2\theta\sec^2\alpha)^{\frac{1}{2}}.$$

Ex. 2. Find the equation to the osculating sphere at the point (1, 2, 3) on the curve

$$x=2t+1,\quad y=3t^2+2,\quad z=4t^3+3.$$

Ans. $3x^2+3y^2+3z^2-6x-16y-18z+50=0.$

Ex. 3. Find equations to represent the osculating circle at (1, 2, 3) of the curve in the last example.

Ans. The equation to the sphere and $z=3$.

Ex. 4. Prove that at the origin the osculating sphere of the curve

$$x=a_1t^3+3b_1t^2+3c_1t,\quad y=a_2t^3+3b_2t^2+3c_2t,\quad z=a_3t^3+3b_3t^2+3c_3t,$$

is given by
$$\begin{vmatrix} x^2+y^2+z^2, & 2x, & 2y, & 2z \\ 9(b_1c_1+b_2c_2+b_3c_3), & a_1, & a_2, & a_3 \\ 3(c_1^2+c_2^2+c_3^2), & 2b_1, & 2b_2, & 2b_3 \\ 0, & c_1, & c_2, & c_3 \end{vmatrix}=0.$$

Ex. 5. Find the curvature and torsion of the spherical indicatrix of the tangents.

The direction-cosines of the tangent are l_2, m_2, n_2, (§ 197), and if $\delta\alpha$ is an infinitesimal arc, $\text{Lt}\,\frac{\delta\alpha}{\delta\psi}=1.$

Hence, if the curvature is $\frac{1}{\rho_0}$,

$$\frac{1}{\rho_0^2}=\Sigma\left(\frac{dl_2}{d\alpha}\right)^2, \text{ (§ 199)},$$

$$=\Sigma\left(\frac{dl_2}{d\psi}\right)^2=\rho^2\Sigma\left(\frac{dl_2}{ds}\right)^2,$$

$$=\rho^2\Sigma\left(\frac{l_1}{\rho}+\frac{l_3}{\sigma}\right)^2=\frac{\rho^2+\sigma^2}{\sigma^2}.$$

If the torsion is $\frac{1}{\sigma_0}$, $\sigma_0^2=\frac{1-\rho_0^2}{\left(\frac{d\rho_0}{d\alpha}\right)^2}$, as in Ex. 1.

Whence we easily find $$\sigma_0=\frac{\rho^2+\sigma^2}{\rho(\rho\sigma'-\rho'\sigma)},$$

where $$\rho'\equiv\frac{d\rho}{ds} \text{ and } \sigma'\equiv\frac{d\sigma}{ds}.$$

Ex. 6. Prove that the radii of curvature and torsion of the spherical indicatrix of the binormals are $\dfrac{\rho}{\sqrt{\rho^2+\sigma^2}}$ and $\dfrac{\rho^2+\sigma^2}{\sigma(\sigma\rho'-\sigma'\rho)}$.

Ex. 7. A curve is drawn on a right circular cone, semivertical angle α, so as to cut all the generators at the same angle β. Shew that its projection on a plane at right angles to the axis is an equiangular spiral, and find expressions for its curvature and torsion.

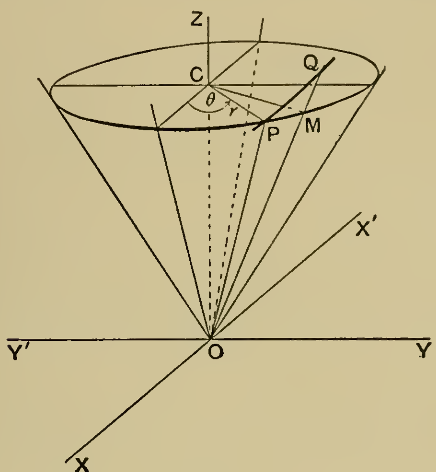

FIG. 57.

Take the vertex of the cone as origin and the axis as z-axis. Let **C**, fig. 57, be the projection of **P**, the point considered, on the axis, and **CP** and **OP** have measures r and R respectively. Then if **CP** makes an angle θ with **OX**, r, θ are the polar coordinates of the projection of **P** on any plane at right angles to the axis.

From fig. 58 we obtain

$$\left.\begin{aligned} dr&=d\mathrm{R}\sin\alpha=dz\tan\alpha,\\ d\mathrm{R}&=ds\cos\beta=r\,d\theta\cot\beta.\end{aligned}\right\} \quad\text{.....................(1)}$$

Whence $$\frac{dr}{r}=\cot\beta\sin\alpha\,d\theta,$$

which is the differential equation to the projection and has as integral

$$r=Ae^{k\theta},$$

where $k\equiv\cot\beta\sin\alpha$ and A is arbitrary.

Again, from (1), $\frac{dz}{ds}=\cos\alpha\cos\beta$, and therefore the tangent to the curve makes a constant angle γ with the z-axis such that

$$\cos\gamma=\cos\alpha\cos\beta.$$

We have therefore $\qquad \frac{d^2z}{ds^2}=0, \quad \sigma=\rho\tan\gamma.$

Since $\qquad \frac{d^2z}{ds^2}=0, \quad \frac{1}{\rho^2}=\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2.$

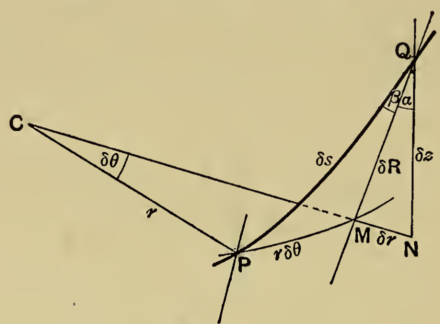

FIG. 58.

Now $\qquad x=r\cos\theta, \quad y=r\sin\theta,$

and, using dashes to denote differentiation with respect to s, by (1),

$$r'=\sin\alpha\cos\beta, \text{ and } r\theta'=\sin\beta.$$

Therefore

$$x'=r'\cos\theta-\sin\beta\sin\theta, \qquad y'=r'\sin\theta+\sin\beta\cos\theta;$$

$$x''=-(r'\sin\theta+\sin\beta\cos\theta)\theta', \quad y''=(r'\cos\theta-\sin\beta\sin\theta)\theta';$$

$$\frac{1}{\rho^2}=\theta'^2(r'^2+\sin^2\beta)=\frac{\sin^2\beta(1-\cos^2\alpha\cos^2\beta)}{r^2}.$$

Hence $\qquad \rho=\frac{r}{\sin\beta\sin\gamma}, \quad \text{and} \quad \sigma=\rho\tan\gamma=\frac{r}{\sin\beta\cos\gamma}.$

Ex. 8. Deduce equations (1), Ex. 7, by differentiating the equations

$$x^2+y^2=z^2\tan^2\alpha, \quad x^2+y^2+z^2=R^2, \quad x^2+y^2=r^2,$$

and applying $\qquad xx'+yy'+zz'=R\cos\beta.$

Ex. 9. The principal normals to a given curve are also principal normals to another curve. Prove that the distance between corresponding points of the curves is constant, that the tangents at corresponding points are inclined at a constant angle, and that there must be a linear relation between the curvature and torsion of the given curve.

If O, (x, y, z) is a point on the given curve, O′, the corresponding point on the second curve has coordinates given by

$$\xi=x+l_2r, \quad \eta=y+m_2r, \quad \zeta=z+n_2r,$$

where OO′ is of length r. If we take O for origin and OT, OP, OB the tangent, principal normal, and binormal as coordinate axes,

$$l_1=1,\ m_1=n_1=0\ ;\quad l_2=n_2=0,\ m_2=1\ ;\quad l_3=m_3=0,\ n_3=1\ ;$$

and O′ is $(0, r, 0)$. The tangent to the second curve is at right angles to OP or OO′, and therefore $\dfrac{d\eta}{ds}=0$; *i.e.* $m_1-r\left(\dfrac{m_1}{\rho}+\dfrac{m_3}{\sigma}\right)+m_2\dfrac{dr}{ds}=0.$

Hence, since $m_1=m_3=0$, $\dfrac{dr}{ds}=0$, and r is constant.

Again, if $\xi'\equiv\dfrac{d\xi}{ds}$, etc., we have

$$\xi'=l_1+l_2\frac{dr}{ds}-r\left(\frac{l_1}{\rho}+\frac{l_3}{\sigma}\right),\ \text{etc.}\ ;$$

therefore, at the origin,

$$\xi'=(1-r/\rho),\quad \eta'=0,\quad \zeta'=-r/\sigma,$$

and the tangent to the second curve makes an angle θ with OT such that $\tan\theta=\dfrac{r/\sigma}{1-r/\rho}$.

$$\xi''=\frac{l_1 r\rho'}{\rho^2}+l_2\left(\frac{1}{\rho}-\frac{r}{\rho^2}-\frac{r}{\sigma^2}\right)+\frac{l_3 r\sigma'}{\sigma^2},\ \text{etc.}\ ;$$

therefore, at the origin,

$$\xi''=\frac{r\rho'}{\rho^2},\quad \eta''=\frac{1}{\rho}-\frac{r}{\rho^2}-\frac{r}{\sigma^2},\quad \zeta''=\frac{r\sigma'}{\sigma^2}.$$

But the binormal to the second curve is at right angles to OP, and therefore

$$\zeta'\xi''-\zeta''\xi'=0,$$

or

$$\frac{\dfrac{r\rho'}{\rho^2}}{1-\dfrac{r}{\rho}}=-\frac{\dfrac{r\sigma'}{\sigma^2}}{\dfrac{r}{\sigma}}.$$

Integrating, we obtain

$$\log A\left(1-\frac{r}{\rho}\right)=\log\frac{r}{\sigma},$$

where A is an arbitrary constant,

or

$$A\left(1-\frac{r}{\rho}\right)-\frac{r}{\sigma}=0,$$

and thus there is a linear relation between the curvature and torsion.

Again, $\tan\theta=\dfrac{r/\sigma}{1-r/\rho}=A$, and therefore θ is constant.

This problem was first investigated by Bertrand, and curves which satisfy the conditions are on that account called **Bertrand curves.**

Ex. 10. A curve is projected on a plane the normal to which makes angles α and β with the tangent and binormal. If ρ_1 is the radius of curvature of the projection, prove that $\rho=\dfrac{\rho_1\cos\beta}{\sin^3\alpha}$.

Let P be a point of the curve and Q and R the points of the curve distant δs from P. Then, if the area of the triangle PQR is denoted by Δ, $\frac{1}{\rho}=\text{Lt}\frac{2\Delta}{\delta s^3}$. Similarly for the projection $\frac{1}{\rho_1}=\text{Lt}\frac{2\Delta'}{\delta s'^3}$, where Δ', $\delta s'$ are the projections of Δ and δs. But $\Delta'=\Delta\cos\beta$, and $\delta s'=\delta s\sin\alpha$, whence the result.

205. Geometrical investigation of curvature and torsion. The following geometrical investigation of the curvature and torsion of a curve is instructive.

Let A_1, A_2, A_3, ... , (fig. 59), be consecutive vertices of an equilateral polygon inscribed in a given curve, and let

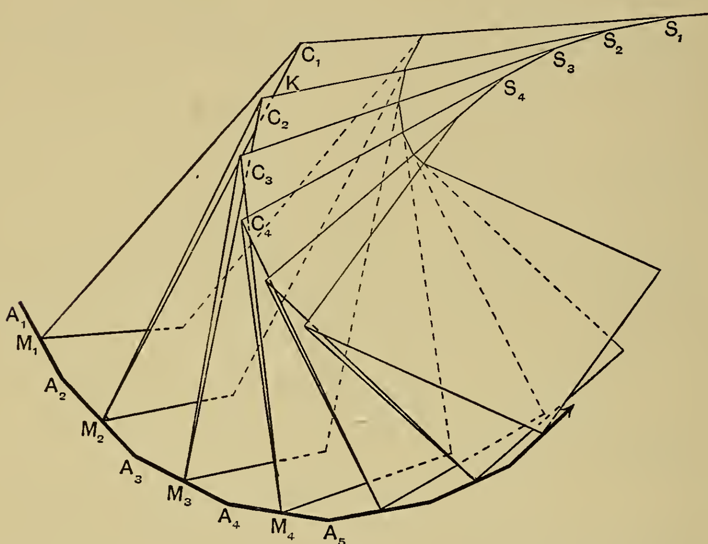

FIG. 59.

M_1, M_2, M_3, ... be the mid-points of the sides. Planes $M_1C_1S_1$, $M_2C_2S_2$, ... are drawn through M_1, M_2, ... normal to the sides. M_1C_1, M_2C_1 are the lines of intersection of the planes $M_1C_1S_1$, $M_2C_2S_2$ and the plane $A_1A_2A_3$; similarly, M_2C_2 and M_3C_2 lie in the plane $A_2A_3A_4$, and so on.

Then C_1 is the centre of the circle through the points A_1, A_2, A_3, and its limiting position when A_2 and A_3 tend to A_1 is the centre of curvature at A_1. Let ρ denote the

radius of the circle of curvature at A_1. From the cyclic quadrilateral $C_1M_1A_2M_2$,

$$\frac{A_1A_2}{2\sin\frac{1}{2}M_1C_1M_2}=C_1A_2.$$

But since the limiting positions of A_1A_2 and A_2A_3 are tangents,

$$\mathrm{Lt}\,\frac{A_1A_2}{2\sin\frac{1}{2}M_1C_1M_2}=\frac{ds}{d\psi}.$$

Therefore, since the limiting value of C_1A_2 is ρ,

$$\rho=\frac{ds}{d\psi},$$

and the curvature $=\dfrac{d\psi}{ds}=\dfrac{1}{\rho}$, where ρ is the radius of the circle of curvature.

Since the planes $M_1C_1S_1$, $M_2C_1S_1$ are at right angles to the plane $A_1A_2A_3$, their line of intersection C_1S_1 is normal to the plane $A_1A_2A_3$. Therefore, in the limit, C_1S_1 is parallel to the binormal at A_1. But since C_1S_1 is the locus of points equidistant from the points A_1, A_2, A_3, and C_2S_1 is the locus of points equidistant from A_2, A_3, A_4, S_1 is the centre of the sphere through A_1, A_2, A_3, A_4, and the limiting position of S_1 is the centre of spherical curvature at A_1. Therefore, the centre of spherical curvature lies on the line drawn through the centre of circular curvature parallel to the binormal.

Since the limiting positions of C_1S_1 and C_2S_1 are parallel to consecutive binormals, we may denote the angle $C_1S_1C_2$ by $\delta\tau$. If C_1M_2 and C_2S_1 intersect at K, then $C_1K=C_1S_1\delta\tau$. But C_1K differs from $C_1M_1-C_2M_2$ by an infinitesimal of higher order, and therefore

$$\mathrm{Lt}\,C_1S_1=\mathrm{Lt}\,\frac{C_1K}{\delta\tau}=\mathrm{Lt}\,\frac{C_1M_1-C_2M_2}{\delta\tau}=\frac{d\rho}{d\tau}.$$

Hence, if R is the radius of spherical curvature at A_1,

$$R^2=\mathrm{Lt}(M_1C_1{}^2+C_1S_1{}^2)=\rho^2+\left(\frac{d\rho}{d\tau}\right)^2.$$

By our convention of § 193, the positive direction of the binormal is that of S_1C_1. In our figure the curve is

dextrorsum, and $\delta\tau$ is therefore negative. Also $\delta\rho$ is negative, so that $\delta\rho/\delta\tau$ is positive. Hence, if the coordinates of A_1 are x, y, z, and those of the limiting position of S_1 are x_0, y_0, z_0,

$$x_0 - x = \text{projection of } M_1C_1 + \text{projection of } C_1S_1 \text{ on } OX,$$

$$= \rho l_2 - \frac{d\rho}{d\tau} l_3,$$

or
$$x_0 = x + \rho l_2 - \frac{d\rho}{d\tau} l_3.$$

Similarly,

$$y_0 = y + \rho m_2 - \frac{d\rho}{d\tau} m_3, \quad z_0 = z + \rho n_2 - \frac{d\rho}{d\tau} n_3.$$

The points S_1, S_2, S_3, ... are consecutive points of a curve which is the locus of the centres of spherical curvature, and S_1S_2, S_2S_3, ... are ultimately tangents to that locus. The plane $S_1S_2S_3$ or $M_3C_2S_1$ is ultimately an osculating plane to the locus, and hence the osculating planes of the locus are the normal planes of the curve. Therefore, if $\delta\psi$, $\delta\tau$ are the angles between adjacent tangents and binormals to the curve, and $\delta\psi_1$, $\delta\tau_1$ are the angles between adjacent tangents and binormals to the locus,

$$\text{Lt}\,\frac{\delta\psi_1}{\delta\tau} = 1, \quad \text{and} \quad \text{Lt}\,\frac{\delta\tau_1}{\delta\psi} = 1.$$

Hence, if infinitesimal arcs of the curve and locus are denoted by δs and δs_1, and the curvature and torsion of the locus by $1/\rho_1$ and $1/\sigma_1$,

$$\rho\rho_1 = \text{Lt}\,\frac{\delta s}{\delta\psi}\cdot\frac{\delta s_1}{\delta\psi_1} = \text{Lt}\,\frac{\delta s}{\delta\tau}\cdot\frac{\delta s_1}{\delta\tau_1} = \sigma\sigma_1.$$

The limiting positions of C_1S_1, C_2S_1, ... are the generators of a ruled surface which is called the **polar developable**. Since C_1S_1 and C_2S_1 are ultimately coincident, the plane $C_1S_1C_2$ touches this surface at all points of the generator C_1S_1, and hence the normal planes to the curve are the tangent planes to the polar developable.

Ex. 1. Shew that if ds_1 is the differential of the arc of the locus of the centres of spherical curvature,

$$ds_1 = \frac{R\,dR}{\sqrt{R^2-\rho^2}}.$$

If (ξ, η, ζ) is a point on the locus, and $\xi' \equiv \frac{d\xi}{ds}$, etc.,

$$\xi = x + \rho l_2 - \sqrt{R^2-\rho^2}\,l_3.$$

Therefore, by Frenet's formulae, and since $\sqrt{R^2-\rho^2} = \sigma\rho'$,

$$\xi' = -l_3 \frac{RR'}{\sqrt{R^2-\rho^2}}.$$

$$\therefore \frac{ds_1}{ds} = \{\Sigma\xi'^2\}^{\frac{1}{2}} = \frac{RR'}{\sqrt{R^2-\rho^2}}.$$

Ex. 2. Obtain the result from fig. 59.

Ex. 3. Prove that $\rho_1 = R\dfrac{dR}{d\rho}$, $\sigma_1 = \dfrac{R\rho}{\sigma}\dfrac{dR}{d\rho}$, and verify that $\rho\rho_1 = \sigma\sigma_1$.

206. Coordinates in terms of s. If the tangent, principal normal and binormal at a given point O of a curve are taken as coordinate axes, and s measures the arc OP, we may express the coordinates of P in terms of s. We have

$$x = f(s) = f(0) + sf'(0) + \frac{s^2}{\underline{|2}}f''(0) + \frac{s^3}{\underline{|3}}f'''(0)\ldots,$$

$$= sx_0' + \frac{s^2}{2}x_0'' + \frac{s^3}{6}x_0''' + \ldots,$$

where x_0', x_0'', x_0''', ... are the values of x', x'', x''', ... at the origin. Similarly,

$$y = sy_0' + \frac{s^2}{2}y_0'' + \frac{s^3}{6}y_0''' + \ldots,$$

$$z = sz_0' + \frac{s^2}{2}z_0'' + \frac{s^3}{6}z_0''' + \ldots.$$

We have therefore to evaluate x_0', y_0', z_0', etc.

Since the tangent is the x-axis,

$$x_0' = 1, \quad y_0' = 0, \quad z_0' = 0.$$

Since the principal normal is the y-axis,

$$\rho x_0'' = 0, \quad \rho y_0'' = 1, \quad \rho z_0'' = 0.$$

Again, by Frenet's formulae,

$$\frac{l_1}{\rho}+\frac{l_3}{\sigma}=-\frac{dl_2}{ds}=-(\rho x'''+\rho' x''),$$

$$\frac{m_1}{\rho}+\frac{m_3}{\sigma}=-(\rho y'''+\rho' y''),\quad \frac{n_1}{\rho}+\frac{n_3}{\sigma}=-(\rho z'''+\rho' z'').$$

Therefore $x_0'''=-\dfrac{1}{\rho^2},\quad y_0'''=-\dfrac{\rho'}{\rho^2},\quad z_0'''=-\dfrac{1}{\rho\sigma}.$

Therefore, as far as the terms in s^3, we have

$$x=s-\frac{s^3}{6\rho^2},$$

$$y=\frac{s^2}{2\rho}-\frac{s^3}{6\rho^2}\rho',$$

$$z=-\frac{s^3}{6\rho\sigma}.$$

Ex. 1. Shew that the curve crosses its osculating plane at each point.

Unless $1/\sigma$ is zero, z changes sign with s. If, at any point, $1/\sigma=0$, the osculating plane is said to be **stationary**.

Ex. 2. Prove that the projection of the curve on the normal plane at O has a cusp at O. What is the shape at O of the projections on the osculating plane and rectifying plane?

Ex. 3. If s^2 and higher powers of s can be rejected, shew that the direction-cosines of the tangent, principal normal, and binormal at P are given by

$$\frac{l_1}{1}=\frac{m_1}{s/\rho}=\frac{n_1}{0},\quad \frac{l_2}{-s/\rho}=\frac{m_2}{1}=\frac{n_2}{-s/\sigma},\quad \frac{l_3}{0}=\frac{m_3}{s/\sigma}=\frac{n_3}{1}.$$

In the following examples O and P are adjacent points of a curve, and the arc OP is of length s.

Ex. 4. The angle between the principal normals at O and P is

$$s(\rho^{-2}+\sigma^{-2})^{\frac{1}{2}}.$$

Ex. 5. The shortest distance between the principal normals at O and P is of length $\dfrac{s\rho}{\sqrt{\rho^2+\sigma^2}}$, and it divides the radius of the circle of curvature at O in the ratio $\rho^2:\sigma^2$.

Ex. 6. The angle that the shortest distance between the tangents at O and P makes with the binormal at O is $s/2\sigma$.

Ex. 7. Prove that the shortest distance between the tangents at O and P is $s^3/12\rho\sigma$.

Ex. 8. The osculating spheres at O and P cut at an angle $\dfrac{s\rho}{\sigma R}\dfrac{dR}{d\rho}$.

Examples XI.

1. Shew that the feet of the perpendiculars from the origin to the tangents to the helix $x=a\cos\theta$, $y=a\sin\theta$, $z=c\theta$, lie on the hyperboloid $x^2/c^2+y^2/c^2-z^2/a^2=a^2/c^2$.

2. A curve is drawn on the helicoid $z=c\tan^{-1}y/x$ so as always to cut the generators at a constant angle α. Shew that by properly choosing the starting point it may be made to coincide with the intersection of the helicoid with the cylinder $2r=c(e^{\theta\cot\alpha}-e^{-\theta\cot\alpha})$, r, θ being ordinary polar coordinate. Find the equations to the principal normal at any point.

3. Find $f(\theta)$ so that $x=a\cos\theta$, $y=a\sin\theta$, $z=f(\theta)$ determine a plane curve.

4. If the osculating plane at every point of a curve pass through a fixed point, the curve must be plane. Hence prove that the curves of intersection of the surfaces $x^2+y^2+z^2=a^2$, $2(x^4+y^4+z^4)=a^4$ are circles of radius a.

5. A right helix of radius a and slope α has four-point contact with a given curve at the point where its curvature and torsion are $1/\rho$ and $1/\sigma$. Prove that

$$a=\frac{\rho\sigma^2}{\rho^2+\sigma^2} \quad \text{and} \quad \tan\alpha=\frac{\rho}{\sigma}.$$

6. For the curve $x=a\tan\theta$, $y=a\cot\theta$, $z=\sqrt{2}a\log\tan\theta$,

$$\rho=-\sigma=\frac{2\sqrt{2}a}{\sin^2 2\theta}.$$

7. Shew that the osculating plane at (x, y, z) on the curve

$$x^2+2ax=y^2+2by=z^2+2cz$$

has equation

$$(b^2-c^2)(\xi-x)(x+a)^3+(c^2-a^2)(\eta-y)(y+b)^3+(a^2-b^2)(\zeta-z)(z+c)^3=0.$$

8. Shew that there are three points on the cubic

$$x=a_1t^3+3b_1t^2+3c_1t+d_1 \quad y=a_2t^3+3b_2t^2+3c_2t+d_2,$$
$$z=a_3t^3+3b_3t^2+3c_3t+d_3,$$

the osculating planes at which pass through the origin, and that they lie in the plane

$$3\begin{vmatrix} x, & y, & z \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix} = \begin{vmatrix} x, & y, & z \\ a_1, & a_2, & a_3 \\ d_1, & d_2, & d_3 \end{vmatrix}.$$

9. If ρ, ρ_1, ρ_2, ρ_3 are the radii of curvature of a curve and its projections on the coordinate planes, and α, β, γ are the angles that the tangent makes with the coordinate axes, prove that

$$\frac{\sin^6\alpha}{\rho_1^2}+\frac{\sin^6\beta}{\rho_2^2}+\frac{\sin^6\gamma}{\rho_3^2}=\frac{1}{\rho^2},$$

$$\frac{\sin^3\alpha\cos\alpha}{\rho_1}+\frac{\sin^3\beta\cos\beta}{\rho_2}+\frac{\sin^3\gamma\cos\gamma}{\rho_3}=0.$$

10. Prove that at the point of intersection of the surfaces $x^2+y^2=z^2$, $z=a\tan^{-1}y/x$, where $y=x\tan\theta$, the radius of curvature of the intersection is $\dfrac{a(2+\theta^2)^{\frac{3}{2}}}{(8+5\theta^2+\theta^4)^{\frac{1}{2}}}$.

11. A catenary, constant c, is wrapped round a right circular cylinder, radius a, so that its axis lies along a generator. Shew that the osculating plane at a point of the curve so formed cuts the tangent plane to the cylinder at the point at a constant angle $\tan^{-1}c/a$.

$$\left(x=a\cos\theta,\ y=a\sin\theta,\ z=c\cosh\frac{a\theta}{c}.\right)$$

Prove also that $\rho=\dfrac{az^2}{c\sqrt{a^2+c^2}}$, $\sigma=\dfrac{az^2}{c\sqrt{z^2-c^2}}$.

12. If a curve is drawn on a right circular cylinder so that its osculating plane at any point makes a constant angle with the tangent plane at the point to the cylinder, then when the cylinder is developed into a plane, the curve develops into a catenary.

13. For the helix prove the following properties: the normal at P to the cylinder is the principal normal at P to the helix; the binormal at P makes a constant angle with the axis of the cylinder; the locus of the centre of circular and spherical curvature is a helix; if P′ is the centre of circular curvature at P, P is the centre of circular curvature at P′ for the locus.

14. A curve is drawn on a sphere of radius a, and the principal normal at a point P makes an angle θ with the radius of the sphere to P. Prove that $\rho=a\cos\theta$, $\dfrac{1}{\sigma}=\pm\dfrac{d\theta}{ds}$.

15. If O, P are adjacent points of a curve and the arc OP$=s$, shew that the difference between the chord OP and the arc OP is $s^3/24\rho^2$, powers of s higher than the third being neglected.

16. Prove that

$$x'''^2+y'''^2+z'''^2=\frac{1}{\rho^2\sigma^2}+\frac{1+\rho'^2}{\rho^4},$$

where dashes denote differentiation with respect to s.

17. If from any point of a curve equal infinitesimal arcs of length s are measured along the curve and the circle of curvature, the distance between their extremities is $s^3R/6\rho^2\sigma$.

18. The shortest distance between consecutive radii of spherical curvature divides the radius in the ratio $\sigma^2:\rho^2\left(\dfrac{dR}{d\rho}\right)^2$.

19. A curve is drawn on the paraboloid $x^2+y^2=2pz$ making a constant angle α with the z-axis. Shew that its projection on the plane $z=0$ is given by

$$\theta=\frac{\sqrt{r^2-a^2}}{a}-\cos^{-1}\frac{a}{r},$$

where $a\equiv p\cot\alpha$, and find expressions for its curvature and torsion.

20. A curve is drawn on a sphere, radius a, so as to cut all the meridians at the same angle α. Shew that if θ is the latitude of any point of the curve,

$$\rho=\frac{a\cos\theta}{\sqrt{1-\sin^2\theta\cos^2\alpha}},\quad \sigma=\frac{a^3\tan\alpha}{(a^2-\rho^2\cos^2\alpha)}.$$

21. A point Q is taken on the binormal at a variable point P of a curve of constant torsion $1/\sigma$ so that PQ is of constant length c. Shew that the binormal of the curve traced by Q makes an angle $\tan^{-1}c\rho/\sigma\sqrt{c^2+\sigma^2}$ with PQ.

22. A point moves on a sphere of radius a so that its latitude is equal to its longitude. Prove that at (x, y, z)

$$\rho=\frac{(2a^2-z^2)^{\frac{3}{2}}}{a(8a^2-3z^2)^{\frac{1}{2}}},\quad \sigma=\frac{8a^2-3z^2}{6(a^2-z^2)^{\frac{1}{2}}}.$$

23. A curve is drawn on a right cone so as to cut all the generators at the same angle. Shew that the locus of its centres of spherical curvature satisfies the same conditions.

24. A curve is drawn on a paraboloid of revolution, latus rectum c, so as to make an angle $\pi/4$ with the meridians. Investigate the curvature and torsion at any point in the forms

$$c^2/\rho^2=\tan^2\phi(1+3\sin^2\phi-\sin^4\phi-\sin^6\phi),$$
$$c^3/\rho^2\sigma=\sin\phi\tan^2\phi(1+4\sin^2\phi-6\sin^4\phi+4\sin^6\phi+\sin^8\phi),$$

ϕ being the angle which the tangent to the meridian through the point makes with the axis.

25. The normal plane at any point to the locus of the centres of circular curvature of any curve bisects the radius of spherical curvature at the corresponding point of the given curve.

26. A curve is drawn on a right circular cone of semivertical angle α so as to cut all the generating lines at an angle β. The cone is then developed into a plane. Shew that

$$\rho:\rho_0=\sin\alpha:\sqrt{\sin^2\alpha\cos^2\beta+\sin^2\beta},$$

where ρ, ρ_0 are the radii of curvature at a point of the original curve and of the developed curve respectively.

27. The coordinates of a point of a curve are functions of a parameter t. Prove that the line drawn through any point (x, y, z) of the curve, with direction-cosines proportional to $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$, lies in the osculating plane at the point and makes with the principal normal an angle $\tan^{-1}\left(\rho\frac{d^2t}{ds^2}\cdot\frac{ds}{dt}\right)$.

28. A curve is drawn on a cylinder of radius a and the cylinder is developed into a plane. If ρ be the radius of curvature of the curve and ρ_1 the radius of curvature of the developed curve at corresponding points, $\frac{1}{\rho^2}-\frac{1}{\rho_1{}^2}=\frac{\sin^4\phi}{a^2}$, where ϕ is the angle that the tangent to the curve makes with the generator of the cylinder through the point.

29. A length equal to the radius of torsion, σ, being marked off along the binormals to a curve of constant torsion, prove that ρ_0, the radius of curvature of the locus so formed, is given by $\dfrac{4\rho^2\sigma^2}{\rho_0^{\,2}}=\rho^2+2\sigma^2$. Prove also that the direction-cosines of the binormal to the locus are

$$\sqrt{2}\rho_0/4\sigma, \quad \sqrt{2}\rho_0/4\sigma, \quad \sqrt{2}\rho_0/2\rho.$$

30. A length c is measured along the principal normals to a curve. Shew that the radius of curvature, ρ_0, of the locus is given by

$$\frac{1}{\rho_0^{\,2}}=\left\{\frac{c\rho^2-\sigma^2(\rho-c)}{c^2\rho^2+\sigma^2(\rho-c)^2}\right\}^2+\frac{c^2\rho^2\sigma^2\{c\sigma\rho'+\rho(\rho-c)\sigma'\}^2}{\{c^2\rho^2+\sigma^2(\rho-c)^2\}^3}.$$

31. With any point of a curve as vertex is described the right circular cone having closest contact at the point. Shew that its axis lies in the plane containing the binormal and tangent to the curve and that its semivertical angle is $\tan^{-1}3\sigma/4\rho$.

32. P is a variable point of a given curve and A a fixed point so that the arc $\mathsf{AP}=s$. A point Q is taken on the tangent at P so that the tangent at Q to the locus of Q is at right angles to the tangent at P to the curve. Prove that $\mathsf{PQ}=a-s$, where a is an arbitrary constant. Prove also that if λ_1, μ_1, ν_1; λ_2, μ_2, ν_2; λ_3, μ_3, ν_3 are the direction-cosines of the tangent, principal normal and binormal to the locus,

$$\lambda_1=l_2, \text{ etc.,} \quad \lambda_2=\frac{l_1\sigma+l_3\rho}{\sqrt{\rho^2+\sigma^2}}, \text{ etc;} \quad \lambda_3=\frac{l_1\rho-l_3\sigma}{\sqrt{\rho^2+\sigma^2}}, \text{ etc.,}$$

and that its radii of curvature and torsion are

$$\frac{\sigma(a-s)}{\sqrt{\rho^2+\sigma^2}}, \quad \frac{(\rho^2+\sigma^2)(a-s)}{\rho(\sigma\rho'-\rho\sigma')}.$$

33. Prove that the radius of curvature, ρ_1, of the locus of the centres of circular curvature is given by

$$\frac{1}{\rho_1^{\,2}}=\frac{\rho'^2\sigma^2(\rho^2+\rho'^2)+\rho^2(\rho^2+2\rho'^2-\rho\rho'')^2}{\rho^2(\rho^2+\rho'^2)^3},$$

where $$\rho'\equiv\frac{d\rho}{d\tau}.$$

34. With any point of a curve as vertex is described the paraboloid of revolution having closest contact at the point. Prove that its latus rectum is equal to the diameter of the osculating sphere.

CHAPTER XV.

ENVELOPES.

207. Envelope of system of surfaces whose equation contains one parameter. The equation

$$f(x, y, z, a)=0,$$

where a is an arbitrary parameter, can be made to represent the different members of a system of surfaces by assigning different values to a. The curve of intersection of the surfaces corresponding to the values α, $\alpha+\delta\alpha$, is given by

$$f(x, y, z, \alpha)=0, \quad f(x, y, z, \alpha+\delta\alpha)=0,$$

or by

$$f(x, y, z, \alpha)=0, \quad \frac{f(x, y, z, \alpha+\delta\alpha)-f(x, y, z, \alpha)}{\delta\alpha}=0,$$

that is, by

$$f(x, y, z, \alpha)=0, \quad \frac{\partial}{\partial\alpha}f(x, y, z, \alpha+\theta\,\delta\alpha)=0,$$

where θ is a proper fraction.

Hence, as $\delta\alpha$ tends to zero, the curve tends to a limiting position given by

$$f(x, y, z, \alpha)=0, \quad \frac{\partial}{\partial\alpha}f(x, y, z, \alpha)=0.$$

This limiting position is called the **characteristic** corresponding to the value α. The locus of the characteristics for all values of a is the **envelope** of the system of surfaces. Its equation is obtained by eliminating a between the two equations

$$f(x, y, z, a)=0, \quad \frac{\partial}{\partial a}f(x, y, z, a)=0.$$

Ex. Find the envelope of spheres of constant radius whose centres lie on OX.

The equation to the spheres of the system is

$$(x-a)^2+y^2+z^2=r^2,$$

where a is an arbitrary parameter and r is constant. The characteristic corresponding to $a=\alpha$ is the great circle of the sphere

$$(x-\alpha)^2+y^2+z^2=r^2$$

which lies in the plane $x=\alpha$, and the envelope is the cylinder

$$y^2+z^2=r^2.$$

208. *The envelope touches each surface of the system at all points of the corresponding characteristic.*

Consider the surface given by $a=\alpha$. The equations to the characteristic are

$$f(x, y, z, \alpha)=0, \quad f_\alpha(x, y, z, \alpha)=0.$$

The equation to the envelope may be obtained by eliminating α between the equations to the characteristic, and this may be effected by solving the equation $f_\alpha(x, y, z, \alpha)=0$ for α, and substituting in $f(x, y, z, \alpha)=0$. Thus, we may regard the equation $f(x, y, z, \alpha)=0$, where α is a function of x, y, z given by $f_\alpha(x, y, z, \alpha)=0$ as the equation to the envelope. The tangent plane at (x, y, z) to the envelope is therefore

$$\xi\left(f_x+f_\alpha\frac{\partial\alpha}{\partial x}\right)+\eta\left(f_y+f_\alpha\frac{\partial\alpha}{\partial y}\right)+\zeta\left(f_z+f_\alpha\frac{\partial\alpha}{\partial z}\right)+t\left(f_t+f_\alpha\frac{\partial\alpha}{\partial t}\right)=0,$$

where t is introduced to make the equations $f=0, f_\alpha=0$ homogeneous. But at any point of the characteristic $f_\alpha=0$, and the above equation becomes

$$\xi f_x+\eta f_y+\zeta f_z+tf_t=0,$$

which represents the tangent plane at (x, y, z) to the surface $f=0$. Hence the envelope and surface have the same tangent plane at any point of the characteristic.

At any point of the characteristic corresponding to $a=\alpha$, we have

$$f_xdx+f_ydy+f_zdz+f_\alpha d\alpha=0 \quad\text{and}\quad f_\alpha=0,$$

and therefore $$f_xdx+f_ydy+f_zdz=0.$$

But if (x, y, z) is a singular point on the surface $f(x, y, z, \alpha)=0$, $f_x=f_y=f_z=0$, and hence the characteristic passes through the singular point. The locus of the singular points of the surfaces of the system

therefore lies on the envelope. For any point of the locus the coefficients in the equation to the tangent plane to the envelope are all zero, and the proposition thus fails for such points.

Consider, for example, the envelope of the right cones of given semivertical angle α, whose vertices lie upon OX and whose axes are parallel to OZ. The equations to the system and to the envelope are

$$(x-a)^2+y^2=z^2\tan^2\alpha, \quad y^2=z^2\tan^2\alpha.$$

The locus of the singular points of the system is OX, and the tangent planes to the envelope and surfaces are indeterminate at any point of the locus.

209. The edge of regression. The equations to the characteristics corresponding to values α and $\alpha+\delta\alpha$ of a are

$$(f=0)_{a=\alpha}, \quad \left(\frac{\partial f}{\partial a}=0\right)_{a=\alpha};$$

$$(f=0)_{a=\alpha+\delta\alpha}, \quad \left(\frac{\partial f}{\partial a}=0\right)_{a=\alpha+\delta\alpha}.$$

The coordinates of any common point of these characteristics satisfy the four equations, and therefore satisfy the equations

$$(f=0)_{a=\alpha}, \quad \left(\frac{\partial f}{\partial a}=0\right)_{a=\alpha+\theta_1\delta\alpha}, \quad \left(\frac{\partial^2 f}{\partial a^2}=0\right)_{a=\alpha+\theta_2\delta\alpha},$$

where θ_1 and θ_2 are proper fractions. Hence, as $\delta\alpha$ tends to zero, the common points tend to limiting positions given by

$$(f=0)_{a=\alpha}, \quad \left(\frac{\partial f}{\partial a}=0\right)_{a=\alpha}, \quad \left(\frac{\partial^2 f}{\partial a^2}\right)_{a=\alpha}=0. \quad \text{.........(1)}$$

These limiting positions for all values of a lie upon a curve whose equations are obtained by the elimination of α between equations (1). This locus is called the **edge of regression** or **cuspidal edge** of the envelope.

210. Each characteristic touches the edge of regression. We may consider the equations

$$f=0, \quad f_a=0,$$

where a is a function of x, y, z, given by $f_{aa}=0$, to represent two surfaces whose curve of intersection is the edge of regression. The tangent at (x, y, z) to the edge of regression

is the line of intersection of the tangent planes to the surfaces. Its equations are therefore

$$\Sigma\xi\left(f_x+f_a\frac{\partial a}{\partial x}\right)=0, \quad \Sigma\xi\left(f_{xa}+f_{aa}\frac{\partial a}{\partial x}\right)=0.$$

At any point of the edge of regression we have $f_a=0$, $f_{aa}=0$, and the above equations become

$$\Sigma\xi f_x=0, \quad \Sigma\xi f_{xa}=0,$$

which represent the tangent at x, y, z, to the curve $f=0$, $f_a=0$, *i.e.* to a characteristic.

Ex. 1. Find the envelope of the plane $3xt^2-3yt+z=t^3$, and shew that its edge of regression is the curve of intersection of the surfaces $y^2=xz$, $xy=z$.

Ex. 2. Find the envelope of the sphere

$$(x-a\cos\theta)^2+(y-a\sin\theta)^2+z^2=b^2.$$

Ans. $(x^2+y^2+z^2+a^2-b^2)^2=4a^2(x^2+y^2)$.

Ex. 3. The envelope of the surfaces $f(x, y, z, a, b)=0$, where a and b are parameters connected by the equation $\phi(a, b)=0$, is found by eliminating a and b between the equations $f=0$, $\phi=0$, $\dfrac{f_a}{\phi_a}=\dfrac{f_b}{\phi_b}$.

Ex. 4. The envelope of the surfaces $f(x, y, z, a, b, c)=0$, where a, b, c are parameters connected by the equation $\phi(a, b, c)=0$, and f and ϕ are homogeneous with respect to a, b, c, is found by eliminating a, b, c between the equations $f=0$, $\phi=0$, $\dfrac{f_a}{\phi_a}=\dfrac{f_b}{\phi_b}=\dfrac{f_c}{\phi_c}$.

Ex. 5. Find the envelope of the plane $lx+my+nz=0$, where $al^2+bm^2+cn^2=0$. *Ans.* $x^2/a+y^2/b+z^2/c=0$.

Ex. 6. The envelope of the osculating plane of a curve is a ruled surface which is generated by the tangents to the curve, and has the curve for its edge of regression.

The equation to the osculating plane is $\Sigma l_3(\xi-x)=0$, where l_3, m_3, n_3, x, y, z, are functions of s. A characteristic is given by

$$\Sigma l_3(\xi-x)=0, \quad \Sigma l_2(\xi-x)=0, \text{ (Frenet's formulae)},$$

or

$$\frac{\xi-x}{l_1}=\frac{\eta-y}{m_1}=\frac{\zeta-z}{n_1},$$

which represent a tangent to the curve.

A point on the edge of regression is given by

$$\Sigma l_3(\xi-x)=0, \quad \Sigma l_2(\xi-x)=0, \quad \Sigma l_1(\xi-x)=0,$$

whence $\xi=x$, $\eta=y$, $\zeta=z$, and the points of the edge of regression are the points of the curve.

Ex. 7. Prove that the envelope of the normal planes drawn through the generators of the cone $ax^2+by^2+cz^2=0$ is given by

$$a^{\frac{1}{3}}(b-c)^{\frac{2}{3}}x^{\frac{2}{3}}+b^{\frac{1}{3}}(c-a)^{\frac{2}{3}}y^{\frac{2}{3}}+c^{\frac{1}{3}}(a-b)^{\frac{2}{3}}z^{\frac{2}{3}}=0.$$

211. Envelope of a system of surfaces whose equation contains two parameters. The equation

$$f(x, y, z, a, b)=0,$$

where a and b are parameters, may also be taken to represent a system of surfaces. The curve of intersection of the surfaces corresponding to values α, $\alpha+\delta\alpha$ of a, and β, $\beta+\delta\beta$ of b, is given by

$$f(\alpha, \beta)=0, \quad f(\alpha+\delta\alpha, \beta+\delta\beta)=0,$$

or by

$$f(\alpha, \beta)=0,$$

$$\delta\alpha\frac{\partial}{\partial\alpha}f(\alpha+\theta_1\delta\alpha, \beta+\delta\beta)+\delta\beta\frac{\partial}{\partial\beta}f(\alpha, \beta+\theta_2\delta\beta)=0,$$

where θ_1 and θ_2 are proper fractions. If $\delta\beta=\lambda\delta\alpha$, the curve of intersection is given by

$$f(\alpha, \beta)=0, \quad \frac{\partial}{\partial\alpha}f(\alpha+\theta_1\delta\alpha, \beta+\delta\beta)+\lambda\frac{\partial}{\partial\beta}f(\alpha, \beta+\theta_2\delta\beta)=0,$$

and the limiting position as $\delta\alpha$ and $\delta\beta$ tend to zero, by

$$f(\alpha, \beta)=0, \quad \frac{\partial f}{\partial\alpha}+\lambda\frac{\partial f}{\partial\beta}=0.$$

But $\delta\alpha$ and $\delta\beta$ are independent, so that λ can assume any value, and the limiting position of the curve depends on the value of λ and will be different for different values of λ. The limiting positions, however, for all values of λ will pass through the points given by

$$f(\alpha, \beta)=0, \quad \frac{\partial f}{\partial\alpha}=0, \quad \frac{\partial f}{\partial\beta}=0.$$

These are called **characteristic points**, and the locus of the characteristic points is the envelope of the system of surfaces. The equation to the envelope is found by eliminating a and b between the three equations

$$f(x, y, z, a, b)=0, \quad \frac{\partial f(x, y, z, a, b)}{\partial a}=0, \quad \frac{\partial f(x, y, z, a, b)}{\partial b}=0.$$

Consider for example the system of spheres of constant radius whose centres are on the xy-plane. The equation to the system is

$$(x-a)^2+(y-b)^2+z^2=r^2,$$

where a and b are arbitrary parameters, and r is constant. Let

$$\mathsf{P}(\alpha, \beta, 0) \text{ and } \mathsf{P}'(\alpha+\delta\alpha, \beta+\delta\beta, 0)$$

be the centres of two spheres of the system. If the ratio $\delta\beta/\delta\alpha$ remains constant the direction of **PP′** is fixed. The limiting position of the curve of intersection of the spheres as **P′** tends to **P** along the line **PP′** is the great circle of the sphere, centre **P**, which is at right angles to **PP′**. But all the limiting positions pass through the extremities of the diameter through **P**, whose equations are $x=\alpha$, $y=\beta$, and these are the characteristic points. Their locus is the pair of planes $z^2=r^2$.

212. *The envelope touches each surface of the system at the corresponding characteristic points.*

Consider the surface $f(x, y, z, \alpha, \beta)=0$. The characteristic points are given by

$$f=0, \quad f_\alpha=0, \quad f_\beta=0.$$

The equation to the envelope may be obtained by eliminating α and β between these three equations, and this may be effected by solving $f_\alpha=0$, $f_\beta=0$ for α and β and substituting in $f=0$. Hence, we may regard

$$f(x, y, z, \alpha, \beta)=0,$$

where α and β are functions of x, y, z, given by $f_\alpha=0$ and $f_\beta=0$, as the equation to the envelope. The tangent plane at (x, y, z) to the envelope has therefore the equation

$$\Sigma\xi\left(f_x+f_\alpha\frac{\partial\alpha}{\partial x}+f_\beta\frac{\partial\beta}{\partial x}\right)=0.$$

But if (x, y, z) is a characteristic point, $f_\alpha=0$ and $f_\beta=0$, and the equation becomes

$$\Sigma\xi f_x=0,$$

which represents the tangent plane at (x, y, z) to the surface. Therefore the envelope and surface have the same tangent plane at a characteristic point.

Ex. 1. Find the envelope of the plane

$$\frac{x}{a}\cos\theta\sin\phi+\frac{y}{b}\sin\theta\sin\phi+\frac{z}{c}\cos\phi=1.$$

Ans. $x^2/a^2+y^2/b^2+z^2/c^2=1.$

Ex. 2. Find the envelope of the plane

$$\frac{(\mu-\lambda)x}{a}+\frac{(1+\lambda\mu)y}{b}+\frac{(1-\lambda\mu)z}{c}=\mu+\lambda,$$

where λ and μ are parameters.

Ans. $x^2/a^2+y^2/b^2-z^2/c^2=1.$

Ex. 3. Prove that the envelope of the surfaces $f(x, y, z, a, b, c)=0$, where a, b, c are parameters connected by the equation $\phi(a, b, c)=0$, is found by eliminating a, b, c between the equations

$$f=0, \quad \phi=0, \quad \frac{f_a}{\phi_a}=\frac{f_b}{\phi_b}=\frac{f_c}{\phi_c}.$$

Ex. 4. Prove that the envelope of the surfaces $f(x, y, z, a, b, c, d)=0$, where a, b, c, d are parameters connected by the equation $\phi(a, b, c, d)=0$ and f and ϕ are homogeneous with respect to a, b, c, d, is found by eliminating a, b, c, d between the equations $f=0$, $\phi=0$ and

$$\frac{f_a}{\phi_a}=\frac{f_b}{\phi_b}=\frac{f_c}{\phi_c}=\frac{f_d}{\phi_d}.$$

Ex. 5. Find the envelope of the plane $lx+my+nz=p$ when

(i) $p^2=a^2l^2+b^2m^2+c^2n^2$, (ii) $a^2l^2+b^2m^2+2np=0$.

Ans. (i) $x^2/a^2+y^2/b^2+z^2/c^2=1$, (ii) $x^2/a^2+y^2/b^2=2z$.

Ex. 6. Find the envelope of a plane that forms with the (rectangular) coordinate planes a tetrahedron of constant volume $c^3/6$.

Ans. $27xyz=c^3$.

Ex. 7. A plane makes intercepts a, b, c on the axes, so that

$$a^{-2}+b^{-2}+c^{-2}=k^{-2}.$$

Shew that it envelopes a conicoid which has the axes as equal conjugate diameters.

Ex. 8. From a point P on the conicoid $a^2x^2+b^2y^2+c^2z^2=1$, perpendiculars PL, PM, PN are drawn to the coordinate planes. Find the envelope of the plane LMN.

Ans. $(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}+(cz)^{\frac{2}{3}}=2^{\frac{2}{3}}$.

Ex. 9. A tangent plane to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ meets the axes in A, B, C. Shew that the envelope of the sphere OABC is

$$(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}+(cz)^{\frac{2}{3}}=(x^2+y^2+z^2)^{\frac{2}{3}}.$$

RULED SURFACES.

213. Skew surfaces and developable surfaces. If, in the equations to a straight line

$$x=az+\alpha, \quad y=bz+\beta,$$

a, b, α, β are functions of a single parameter t, we can eliminate the parameter between the two equations and thus obtain an equation which represents a surface generated by the line as t varies. The locus is a ruled surface.

The two generators corresponding to values t, $t+\delta t$, of the parameter have for equations

$$\frac{x-\alpha}{a}=\frac{y-\beta}{b}=z, \quad \frac{x-\alpha-\delta\alpha}{a+\delta a}=\frac{y-\beta-\delta\beta}{b+\delta b}=z.$$

Therefore, if d is the shortest distance between them,

$$d=\frac{\delta\alpha\,\delta b-\delta\beta\,\delta a}{\sqrt{\delta a^2+\delta b^2+(a\,\delta b-b\,\delta a)^2}}.$$

But $a+\delta a=a+a'\delta t+a''\dfrac{\delta t^2}{2}\ldots$, etc., where dashes denote differentiation with respect to t. Therefore, if cubes and higher powers of δt are rejected,

$$d=\frac{(\alpha'b'-\beta'a')\delta t+(\alpha'b''+b'\alpha''-a'\beta''-\beta'a'')\delta t^2}{\sqrt{a'^2+b'^2+(ab'-a'b)^2}}.$$

Hence d is an infinitesimal of the same order as δt if $\alpha'b'-\beta'a'\neq 0$. But if $\alpha'b'-\beta'a'=0$, then we have also $\alpha'b''+b'\alpha''-a'\beta''-\beta'a''=0$, and therefore d is at least of the order of δt^3. If, therefore, δt is so small that δt^2 and δt^3 are inappreciable, $d=0$, or the two generators are coplanar. The result may be stated thus: if $\alpha'b'-\beta'a'=0$, consecutive generators of the surface intersect, while if $\alpha'b'-\beta'a'\neq 0$, consecutive generators do not intersect.

If consecutive generators intersect the surface is a **developable** surface, if they do not intersect, it is a **skew** surface. The name developable arises in this way. If A and B, B and C, consecutive generators of a surface, intersect, the plane of B and C may be turned about B until it coincides with the plane of A and B, and thus the whole surface may be developed into a plane without tearing. Clearly cones and cylinders may be so treated, and are therefore developable conicoids. On the other hand the shortest distance between consecutive generators of the same system of a hyperboloid or paraboloid does not vanish, (§ 114), so that the hyperboloid of one sheet and the hyperbolic paraboloid are skew conicoids.

Ex. 1. Shew by means of Exs. 5 and 7, § 206, that the tangents to a curve generate a developable surface and that the principal normals generate a skew surface.

Ex. 2. Shew that the line given by $y=tx-t^3$, $z=t^3y-t^6$ generates a developable surface.

Ex. 3. Shew that the line $x=3t^2z+2t(1-3t^4)$, $y=-2tz+t^2(3+4t^2)$ generates a skew surface.

214. The tangent plane to a ruled surface. We may regard the coordinates of any point on the surface as functions of two variables t and z, given by the equations

$$\xi=az+\alpha,\quad \eta=bz+\beta,\quad \zeta=z.$$

The tangent plane at (t, z) has for equation

$$\begin{vmatrix} \xi-az-\alpha, & \eta-bz-\beta, & \zeta-z \\ a'z+\alpha', & b'z+\beta', & 0 \\ a, & b, & 1 \end{vmatrix}=0,$$

or

$$\begin{vmatrix} \xi-a\zeta-\alpha, & \eta-b\zeta-\beta, & \zeta-z \\ a'z+\alpha', & b'z+\beta', & 0 \\ 0, & 0, & 1 \end{vmatrix}=0,$$

i.e. $$(\xi-a\zeta-\alpha)(b'z+\beta')-(\eta-b\zeta-\beta)(a'z+\alpha')=0. \quad \ldots\ldots(1)$$

This equation clearly represents a plane passing through the line $$\xi=a\zeta+\alpha,\quad \eta=b\zeta+\beta,$$
which is the generator through the point (t, z).

If $a'\beta'-b'\alpha'=0$, or $\dfrac{a'}{b'}=\dfrac{\alpha'}{\beta'}=k$, say, where k is some function of t, equation (1) becomes

$$(\xi-a\zeta-\alpha)-k(\eta-b\zeta-\beta)=0,$$

and is therefore independent of z. The equation then involves t only, and since when t is given, the generator is given, the tangent plane is the same at all points of the generator.

If $a'\beta'-b'\alpha'\neq 0$, the equation (1) contains z and t, so that the plane given by (1) changes position if t is fixed and z varies, or the tangent planes are different at different points of a generator.

Hence the tangent plane to a developable surface is the same at all points of a generator; the tangent planes to a skew surface are different at different points of a generator.

Cor. The equation to the tangent plane to a developable surface contains only one parameter.

215. The generators of a developable surface are tangents to a curve. If the equations

$$x=az+\alpha, \qquad y=bz+\beta;$$
$$x=(a+a'\delta t)z+\alpha+\alpha'\delta t, \quad y=(b+b'\delta t)z+\beta+\beta'\delta t$$

represent consecutive generators of a developable surface, their point of intersection is given by

$$x=\alpha-\frac{a\alpha'}{a'}, \quad y=\beta-\frac{b\beta'}{b'}, \quad z=-\frac{\alpha'}{a'}=-\frac{\beta'}{b'}.$$

These express the coordinates in terms of one parameter t, and hence the locus of the points of intersection of consecutive generators of a developable is a curve.

By differentiation, we obtain

$$x'=\frac{a(\alpha'a''-\alpha''a')}{a'^2}=az', \quad y'=\frac{b(\beta'b''-\beta''b')}{b'^2}=bz',$$

and therefore the tangent to the curve at (x, y, z) has for equations

$$\frac{\xi-x}{a}=\frac{\eta-y}{b}=\zeta-z,$$

or

$$\xi=a\zeta-az+x=a\zeta+\alpha,$$
$$\eta=b\zeta-bz+y=b\zeta+\beta,$$

which represent the generator through (x, y, z).

216. Envelope of a plane whose equation involves one parameter. We have seen that the equation to the tangent plane to a developable involves only one parameter, (§ 214, Cor.). We shall now prove a converse, viz., that the envelope of a plane whose equation involves one parameter is a developable surface. Let

$$u\equiv a\xi+b\eta+c\zeta+d=0,$$

where a, b, c, d are functions of a parameter t, be the equation to the plane. A characteristic is given by

$$u=0, \quad u'=0,$$

and therefore, since u and u' are linear functions of ξ, η, ζ, the characteristics are straight lines and the envelope is a ruled surface. Two consecutive characteristics are given by

$$u=0, \quad u'=0; \quad u+u'\delta t=0, \quad u'+u''\delta t=0;$$

and these clearly lie in the plane $u+u't=0$, and therefore intersect. Hence the envelope is a developable surface.

The edge of regression of the envelope is given by

$$u=0, \quad u'=0, \quad u''=0,$$

and hence, if (x, y, z) is any point on the edge of regression,

$$ax+by+cz+d=0, \quad a'x+b'y+c'z+d'=0,$$
$$a''x+b''y+c''z+d''=0. \quad \text{.................(1)}$$

But the coordinates of any point on the edge of regression are functions of t. Therefore, from (1),

$$ax'+by'+cz'=-(a'x+b'y+c'z+d')=0,$$

and $\quad ax''+by''+cz''=-(a''x+b''y+c''z+d'')=0,$

whence we see that the plane $a\xi+b\eta+c\zeta+d=0$ has three-point contact at (x, y, z) with the edge of regression, or is the osculating plane. Thus a developable surface is the locus of the tangents to, or the envelope of the osculating planes of, its edge of regression.

Ex. 1. Find the equations to the edge of regression of the developable in Ex. 2, § 213.

The point of intersection of consecutive generators is given by

$$x=3t^2, \quad y=2t^3, \quad z=t^6,$$

and these equations may be taken to represent the edge of regression.

Ex. 2. Find the equations to the developable surfaces which have the following curves for edge of regression:

(i) $x=6t, \quad y=3t^2, \quad z=2t^3$;

(ii) $x=a\cos\theta, \quad y=a\sin\theta, \quad z=c\theta$;

(iii) $x=e^t, \quad y=e^{-t}, \quad z=\sqrt{2}t$.

Ans. (i) $(xy-9z)^2=(x^2-12y)(4y^2-3zx)$;

(ii) $x=a(\cos\theta-\lambda\sin\theta), \quad y=a(\sin\theta+\lambda\cos\theta), \quad z=c(\theta+\lambda)$,

where θ and λ are parameters.

(iii) $x=e^t(1+\lambda), \quad y=e^{-t}(1-\lambda), \quad z=\sqrt{2}(t+\lambda)$.

Ex. 3. Find the edge of regression of the envelope of the normal planes of a curve.

A normal plane is given by

$$\Sigma l_1(\xi-x)=0.$$

And by Frenet's formulae, we have for the edge of regression,

$$\Sigma l_1(\xi-x)=0, \quad \Sigma l_2(\xi-x)=\rho, \quad \Sigma l_3(\xi-x)=-\sigma\rho'.$$

Multiplying by l_1, l_2, l_3, and adding, we deduce

$$\xi = x + l_2\rho - l_3\sigma\rho', \quad \text{and similarly,}$$

$$\eta = y + m_2\rho - m_3\sigma\rho', \quad \zeta = z + n_2\rho - n_3\sigma\rho'.$$

Hence, the edge of regression is the locus of the centres of spherical curvature. The envelope is the **polar developable**, (§ 205).

217. The condition that $\zeta = f(\xi, \eta)$ should represent a developable surface. If $\zeta = f(\xi, \eta)$ represents a developable surface, the equation to the tangent plane

$$p\xi + q\eta - \zeta = px + qy - z$$

involves only one parameter. Let $\phi \equiv px + qy - z$. Then, if t is the parameter,

$$p = f_1(t), \quad q = f_2(t), \quad \phi = f_3(t),$$

and hence, by the elimination of t, we can express p and ϕ as functions of q. Now if u and v are functions of x and y the necessary and sufficient condition that u should be a function of v is $\dfrac{\partial(u, v)}{\partial(x, y)} = 0$.*

Therefore for a developable surface,

$$\begin{vmatrix} p_x, & p_y \\ q_x, & q_y \end{vmatrix} = 0; \quad \text{that is,} \quad \begin{vmatrix} r, & s \\ s, & t \end{vmatrix} \equiv rt - s^2 = 0.$$

A necessary condition is therefore $rt - s^2 = 0$.

Again, $$\begin{vmatrix} \phi_x, & \phi_y \\ q_x, & q_y \end{vmatrix} \equiv \begin{vmatrix} rx + sy, & sx + ty \\ s, & t \end{vmatrix} = x \begin{vmatrix} r, & s \\ s, & t \end{vmatrix}.$$

Therefore, if $rt - s^2 = 0$, ϕ is a function of q. Hence, the necessary and sufficient condition is $rt - s^2 = 0$.

---

* This may be proved as follows:

If $u = f(v)$, $\qquad u_x = v_x f'(v)$,

and $\qquad u_y = v_y f'(v)$,

and therefore $\qquad u_x v_y - u_y v_x = 0$.

Hence $\dfrac{\partial(u, v)}{\partial(x, y)} = 0$ is a necessary condition.

It is also sufficient. For if $u_x v_y - u_y v_x = 0$,

$$\frac{u_x}{v_x} = \frac{u_y}{v_y} = \frac{u_x dx + u_y dy}{v_x dx + v_y dy} = \frac{du}{dv}.$$

Therefore $dv = 0$ if $du = 0$, and hence the variation of u depends only on the variation of v, or u is a function of v.

Ex. 1. By considering the value of $rt-s^2$, determine if the surfaces $xyz=a^3$, $xy=(z-c)^2$ are developable.

Ex. 2. Shew that a developable can be found to circumscribe two given surfaces.

The equation to a plane contains three disposable constants, and the conditions of tangency of the plane and the two surfaces give two equations involving the constants. The equation to the plane therefore involves one constant, and the envelope of the plane is the required developable.

Ex. 3. Shew that a developable can be found to pass through two given curves.

Ex. 4. Shew that the developable which passes through the curves $z=0$, $y^2=4ax$; $x=0$, $y^2=4bz$ is the cylinder $y^2=4ax+4bz$.

Ex. 5. Prove that the edge of regression of the developable that passes through the parabolas $z=0$, $y^2=4ax$; $x=0$, $(y-a)^2=4az$ is the curve of intersection of the surfaces

$$(a+y)^2=3a(x+y+z), \quad (a+y)^3=27a^2x.$$

Any plane which touches the first parabola is

$$\lambda z+my=x+am^2,$$

and if it touches the second, $\lambda=m/(1-m)$. Therefore the equation to a plane which touches both is

$$f(m)\equiv am^3-m^2(a+y)+m(x+y+z)-x=0.$$

Eliminate m between

$$f=0, \quad \frac{df}{dm}=0, \quad \frac{d^2f}{dm^2}=0,$$

and the required result is easily obtained.

Ex. 6. Shew that two cones pass through the curves

$$x^2+y^2=4a^2,\ z=0; \quad x=0,\ y^2=4a(z+a);$$

and that their vertices are the points $(2a, 0, -2a)$, $(-2a, 0, -2a)$.

Ex. 7. Shew that the equation to the developable surface which passes through the curves

$$z=0,\ 4a^3y^3=b^2c^2x^2; \quad y=0,\ 4a^3z^3=bc^4x$$

is

$$(a^2yz-bc^2x)^2=4a^2(bzx+ay^2)(c^2y+az^2),$$

and that its edge of regression is the curve of intersection of the conicoids

$$az^2+c^2y=0, \quad a^2yz-bc^2x=0.$$

Ex. 8. Shew that the edge of regression of the developable that passes through the parabolas $x=0$, $z^2=4ay$; $x=a$, $y^2=4az$ is given by

$$\frac{3x}{y}=\frac{y}{z}=\frac{z}{3(a-x)}.$$

Ex. 9. Prove that the edge of regression of the developable that passes through the circles $z=0$, $x^2+y^2=a^2$, $x=0$, $y^2+z^2=b^2$, lies on the cylinder

$$\left(\frac{x}{a^2}\right)^{\frac{2}{3}}-\left(\frac{z}{b^2}\right)^{\frac{2}{3}}=\left(\frac{1}{a^2}-\frac{1}{b^2}\right)^{\frac{1}{3}}.$$

Ex. 10. Prove that the section by the xy-plane of the developable generated by the tangents to the curve

$$x^2+y^2+z^2=r^2, \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$$

is given by

$$z=0, \quad \frac{a^2(a^2+c^2)}{x^2}+\frac{b^2(b^2+c^2)}{y^2}=\frac{(a^2-b^2)^2}{r^2}.$$

Ex. 11. An ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ is surrounded by a luminous ring $x=0$, $y^2+z^2=a^2$. Shew that the boundary of the shadow cast on the plane $z=0$ is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2-c^2}=\frac{a^2}{a^2-c^2}.$$

218. Properties of a generator of a skew surface. If A_1B_1, A_2B_2, A_3B_3 are any three consecutive generators of a skew surface, a conicoid can be described through A_1B_1, A_2B_2, A_3B_3. The conicoid will be a paraboloid if the generators are parallel to the same plane, as in the case of any conoid, otherwise it will be a hyperboloid. If P is any point on A_2B_2, the two planes through P and A_1B_1, A_3B_3 respectively, intersect in a straight line which meets A_1B_1 and A_3B_3 in Q and R, say. Now PQR meets the conicoid in three ultimately coincident points, and therefore is a generator of the conicoid. Hence the plane of A_2B_2 and PQR is tangent plane at P to the conicoid. But PQR also meets the surface in three ultimately coincident points, and therefore is one of the inflexional tangents through P, the other being the generator A_2B_2. Therefore the plane of PQR and A_2B_2 is also the tangent plane to the surface at P. Thus a conicoid can be found to touch a given skew surface at all points of a given generator.

We can deduce many properties of the generators of a skew surface from those of the generators of the hyperboloid. For example, it follows from § 134, Ex. 10, that if two skew surfaces have a common generator they touch at two points of the generator; and from § 113, Ex. 1, the locus of the normals to a skew surface at points of a given generator is a hyperbolic paraboloid.

Since the surface and conicoid have three consecutive generators in common, the shortest distance and angle

between the given generator and a consecutive generator are the same for both. Hence the generator has the same central point and parameter of distribution for the surface and conicoid. Thus it follows that if the tangent planes at P and P′, points of a given generator of a skew surface, are at right angles, and C is the central point,

$$CP \,.\, CP' = -\delta^2,$$

where δ is the parameter of distribution.

The locus of the central points of a system of generators of a skew surface is a curve on the surface which is called a **line of striction.**

Ex. 1. Prove that the paraboloid which touches the helicoid $y/x = \tan z/c$ at all points of the generator $x \sin\theta = y\cos\theta$, $z = c\theta$ is

$$c(x\sin\theta - y\cos\theta) + (z - c\theta)(x\cos\theta + y\sin\theta) = 0.$$

Prove also that the parameter of distribution of any generator is c, and that the line of striction is the z-axis.

Ex. 2. Prove that the conicoid which touches the surface $y^2 z = 4c^2 x$ at all points of the generator $x = z$, $y = 2c$ is $y(x+3z) = 2c(3x+z)$, and that the normals to the surface at points of the generator lie on the paraboloid $z^2 - x^2 = 4c(y-2c)$.

Ex. 3. For the cylindroid $z(x^2+y^2) = 2mxy$, prove that the parameter of distribution of the generator in the plane $x\sin\theta = y\cos\theta$ is $2m\cos 2\theta$.

Ex. 4. If the line $x = az + \alpha$, $y = bz + \beta$, where a, b, α, β are functions of t, generates a skew surface, the parameter of distribution for the generator is

$$\frac{(\alpha' b' - a'\beta')(1 + a^2 + b^2)}{a'^2 + b'^2 + (ab' - a'b)^2}.$$

Ex. 5. If the line $\dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n}$, (where $l^2+m^2+n^2=1$), generates a skew surface, the parameter of distribution for the generator is

$$\begin{vmatrix} d\alpha, & d\beta, & d\gamma \\ dl, & dm, & dn \\ l, & m, & n \end{vmatrix} \div (dl^2 + dm^2 + dn^2).$$

Deduce the condition that the line should generate a developable surface.

Ex. 6. Apply the result of Ex. 5 to shew that the binormals of a given curve of torsion $1/\sigma$ generate a skew surface and that the parameter of distribution for any generator is the corresponding value of σ.

Ex. 7. Prove that the principal normals to any curve generate a skew surface, the line of striction of which intersects the normal at a

distance $\rho\sigma^2/(\rho^2+\sigma^2)$ from the curve, and that if P, Q are any pair of points on a normal such that the tangent planes at P and Q to the surface are at right angles, $CP \cdot CQ = -\rho^4\sigma^2/(\rho^2+\sigma^2)^2$, where C is the point of intersection of the normal and the line of striction.

(Apply § 206, Exs. 4, 5.)

Ex. 8. Shew that a given curve is the line of striction of the skew surface generated by its binormals.

Ex. 9. If the line $x=az+\alpha, \quad y=bz+\beta$ generates a skew surface, the z-coordinate of the point where the line of striction crosses the generator is

$$-\frac{a'\alpha'(1+b^2)-ab(a'\beta'+b'\alpha')+b'\beta'(1+a^2)}{a'^2+b'^2+(ab'-a'b)^2}.$$

Ex. 10. For the skew surface generated by the line

$$x+yt=3t(1+t^2), \quad y+2zt=t^2(3+4t^2),$$

prove that the parameter of distribution of the generator is $\frac{3}{2}(1+2t^2)^2$, and that the line of striction is the curve

$$x=3t, \quad y=3t^2, \quad z=2t^3.$$

Ex. 11. If the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

generates a skew surface, the point of intersection of the line of striction and the generator is

$$(\alpha+lr, \quad \beta+mr, \quad \gamma+nr),$$

where $$r=\frac{\Sigma(mn'-m'n)(n\beta'-m\gamma')}{\Sigma(mn'-m'n)^2}.$$

Ex. 12. Deduce the results of Exs. 7 and 8.

Ex. 13. The line of striction on a hyperboloid of revolution is the principal circular section.

Ex. 14. Shew that the distance measured along the generator

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{z}{c}$$

of the hyperboloid $x^2/a^2+y^2/b^2-z^2/c^2=1$, from the principal elliptic section to the line of striction, is

$$\frac{c^2(a^2-b^2)\sin\theta\cos\theta(a^2\sin^2\theta+b^2\cos^2\theta+c^2)^{\frac{1}{2}}}{b^2c^2\sin^2\theta+c^2a^2\cos^2\theta+a^2b^2}$$

Examples XII.

1. O is a fixed point on the z-axis and P a variable point on the xy-plane. Find the envelope of the plane through P at right angles to PO.

2. O is a fixed point on the z-axis, and a variable plane through O cuts the xy-plane in a line AB. Find the envelope of the plane through AB at right angles to the plane AOB.

3. Find the envelope of a plane that cuts an ellipsoid in a conic so that the cone whose vertex is the centre of the ellipsoid and whose base is the conic is of revolution.

4. Given three spheres S_1, S_2, S_3, S_1 and S_2 being fixed and S_3 variable and with its centre on S_1. Prove that the radical plane of S_3 and S_2 envelopes a conicoid.

5. The envelope of a plane such that the sum of the squares of its distances from n given points is constant, is a central conicoid whose centre is the mean centre of the given points.

6. Prove that the envelope of the polar planes of a given point with respect to the spheres which touch the axes (rectangular) consists of four parabolic cylinders.

7. Prove that sections of an ellipsoid which have their centres on a given line envelope a parabolic cylinder.

8. Any three conjugate diameters of an ellipsoid meet a fixed sphere concentric with the ellipsoid in P, Q, R. Find the envelope of the plane PQR.

9. A plane meets three intersecting straight lines OX, OY, OZ in A, B, C, so that OA . OB and OB . OC are constant. Find its envelope.

10. Through a fixed point O sets of three mutually perpendicular lines are drawn to meet a given sphere in P, Q, R. Prove that the envelope of the plane PQR is a conicoid of revolution.

11. Find the envelope of a plane that cuts three given spheres in equal circles.

12. Find the envelope of planes which pass through a given point and cut an ellipsoid in ellipses of constant area.

13. O is a fixed point and P any point on a given circle. Find the envelope of the plane through P at right angles to PO.

14. Find the envelope of the normal planes to the curve

$$x^2/a^2+y^2/b^2+z^2/c^2=1, \quad x^2+y^2+z^2=r^2.$$

15. The tangent planes at the feet of the normals from (α, β, γ) to the confocals

$$\frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}+\frac{z^2}{c^2-\lambda}=1$$

envelope a developable surface whose equation is

$$(9C-AB)^2=4(3B-A^2)(3AC-B^2),$$

where A, B, C are the coefficients in the equation in t,

$$\frac{\alpha x}{a^2+t}+\frac{\beta y}{b^2+t}+\frac{\gamma z}{c^2+t}=1.$$

16. The normals from O to one of a system of confocals meet it in P, Q, R ; P′, Q′, R′. If the plane PQR is fixed and O and the confocal vary, find the envelope of the plane P′Q′R′.

17. Prove that the polar planes of (ξ, η, ζ) with respect to the confocals to $x^2/a+y^2/b+z^2/c=1$ are the osculating planes of a cubic curve, and that the general surface of the second degree which passes through the cubic is

$$\lambda(\mathsf{R}^2-3\mathsf{Q})+\mu(\mathsf{RQ}-9\mathsf{P})+\nu(\mathsf{Q}^2-3\mathsf{RP})=0,$$

where $\mathsf{P}\equiv abc-bcx\xi-cay\eta-abz\zeta,\quad \mathsf{R}\equiv a+b+c-x\xi-y\eta-z\zeta,$

$$\mathsf{Q}\equiv bc+ca+ab-(b+c)x\xi-(c+a)y\eta-(a+b)z\zeta.$$

18. Shew that the coordinates of a point on the edge of regression of the rectifying developable, *i.e.* the envelope of the rectifying plane, of a curve are given by

$$\xi=x+\frac{l_3-l_1\tan\theta}{\dfrac{d}{ds}(\tan\theta)},\quad \text{etc., etc.,}$$

where $\tan\theta=\rho/\sigma$. Prove also that the direction-cosines of the tangent and principal normal are proportional to $l_1\rho-l_3\sigma$, etc.; $l_3\rho+l_1\sigma$, etc., and that the radii of curvature and torsion are

$$\frac{1}{\dfrac{d}{ds}(\sin\theta)}\,\frac{d}{ds}\left(\frac{1}{\dfrac{d}{ds}(\tan\theta)}\right)\quad\text{and}\quad \rho\,\frac{d}{ds}\left(\frac{1}{\dfrac{d(\tan\theta)}{ds}}\right).$$

19. If the conicoids

$$x^2/a^2+y^2/b^2+z^2/c^2=1,\quad x^2/a_1^{\,2}+y^2/b_1^{\,2}+z^2/c_1^{\,2}=1$$

are confocal and a developable is circumscribed to the first along its curve of intersection with the second, the edge of regression lies on the cone

$$\frac{x^{\frac{2}{3}}a_1^{\frac{2}{3}}(b^2-c^2)^{\frac{2}{3}}}{a^{\frac{2}{3}}}+\frac{y^{\frac{2}{3}}b_1^{\frac{2}{3}}(c^2-a^2)^{\frac{2}{3}}}{b^{\frac{2}{3}}}+\frac{z^{\frac{2}{3}}c_1^{\frac{2}{3}}(a^2-b^2)^{\frac{2}{3}}}{c^{\frac{2}{3}}}=0.$$

20. A developable surface passes through the curves

$$y=0,\ x^2=(a-b)(2z-b);\quad x=0,\ y^2=(a-b)(2z-a);$$

prove that its edge of regression lies on the cylinder

$$x^{\frac{2}{3}}-y^{\frac{2}{3}}+(a-b)^{\frac{2}{3}}=0.$$

21. Shew that the edge of regression of the envelope of the plane

$$\frac{x}{a+\lambda}+\frac{y}{b+\lambda}+\frac{z}{c+\lambda}=1$$

is the cubic curve given by

$$x=\frac{(a+\lambda)^3}{(c-a)(b-a)},\quad y=\frac{(b+\lambda)^3}{(c-b)(a-b)},\quad z=\frac{(c+\lambda)^3}{(a-c)(b-c)}.$$

22. Prove that the developable surface that envelopes the sphere $x^2+y^2+z^2=c^2$ and the hyperboloid $x^2/a^2+y^2/b^2-z^2/c^2=1$ meets the plane $y=0$ in the conic

$$\frac{x^2}{b^2-a^2}+\frac{z^2}{b^2+c^2}=\frac{c^2}{b^2-c^2}.$$

23. A developable surface is drawn through the curves

$$\lambda^2 x^2 + y^2 = \lambda^2, \; z = c; \quad x^2 + y^2 = 1, \; z = -c;$$

shew that its section by the plane $z=0$ is given by $2x = \sin\alpha + \sin\beta$, $2y = \cos\alpha + \lambda\cos\beta$, where $\tan\alpha = \lambda\tan\beta$.

24. If the generator of a skew surface make with the tangent and principal normal of the line of striction angles whose cosines are λ and μ, prove that $\frac{d\lambda}{ds} = \frac{\mu}{\rho}$, where ρ is the radius of curvature of the line of striction.

25. Prove that the line of striction on the skew surface generated by the line

$$\frac{x - a\cos\theta}{\cos\theta\cos\frac{\theta}{2}} = \frac{y - a\sin\theta}{\sin\theta\cos\frac{\theta}{2}} = \frac{z}{\sin\frac{\theta}{2}}$$

is an ellipse in the plane $2y + z = 0$, whose semiaxes are a, $\frac{3a}{5}$, and whose centre is $\left(\frac{-2a}{5}, 0, 0\right)$.

CHAPTER XVI.

CURVATURE OF SURFACES.

219. We now proceed to investigate the curvature at a point on a given surface of the plane sections of the surface which pass through the point. In our investigation we shall make use of the properties of the indicatrix defined in § 184.

If the point is taken as origin, the tangent plane at the origin as xy-plane, and the normal as z-axis, the equations to the surface and indicatrix are

$$2z = rx^2 + 2sxy + ty^2 + \dots,$$

$$z = h, \quad 2h = rx^2 + 2sxy + ty^2.$$

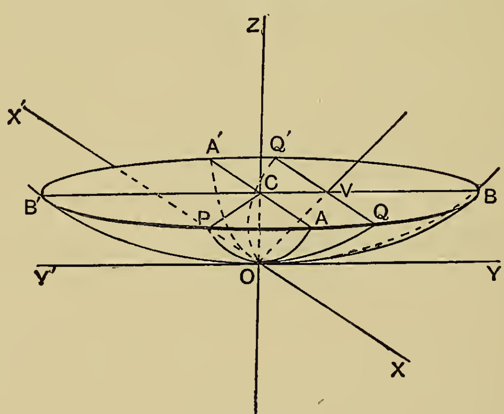

Fig. 60.

220. Curvature of normal sections through an elliptic point. If $rt - s^2 > 0$ the indicatrix is an ellipse, (fig. 60). Let C be its centre, CA and CB its axes, and let CP be any

semidiameter. Then, if ρ is the radius of curvature of the normal section OCP, $\rho = \text{Lt}\,\dfrac{\text{CP}^2}{2\text{OC}}$, and therefore the radii of curvature of normal sections are proportional to the squares of the semidiameters of the indicatrix. The sections OCB, OCA, which have the greatest and least curvature, are called the **principal sections** at O and their radii of curvature are the **principal radii.** If ρ_1, ρ_2 are the principal radii, and CA $= a$, CB $= b$,

$$\rho_1 = \text{Lt}\,\frac{a^2}{2h}, \quad \rho_2 = \text{Lt}\,\frac{b^2}{2h}.$$

If the axes OX and OY are turned in the plane XOY until they lie in the principal sections OCA, OCB respectively, the equations to the indicatrix become

$$z = h, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

or

$$z = h, \quad \frac{x^2}{\rho_1} + \frac{y^2}{\rho_2} = 2h,$$

and the equation to the surface is

$$2z = \frac{x^2}{\rho_1} + \frac{y^2}{\rho_2} + \dots .$$

If CP $= r$, and the normal section OCP makes an angle θ with the principal section OCA, the coordinates of P are $r\cos\theta$, $r\sin\theta$, h. Hence, since P is on the indicatrix,

$$\frac{2h}{r^2} = \frac{\cos^2\theta}{\rho_1} + \frac{\sin^2\theta}{\rho_2};$$

therefore

$$\frac{1}{\rho} = \frac{\cos^2\theta}{\rho_1} + \frac{\sin^2\theta}{\rho_2},$$

where ρ is the radius of curvature of the section OCP.

221. Curvature of normal sections through a hyperbolic point. If $rt - s^2 < 0$, the indicatrix is a hyperbola, (fig. 61). The inflexional tangents are real and divide the surface into two portions such that the concavities of normal sections of the two are turned in opposite directions. If we consider the curvature of a section whose concavity is turned towards the positive direction of the z-axis to be

positive, then the positive radii of curvature are proportional to the squares of semidiameters of the indicatrix

$$z=h,\quad 2h=rx^2+2sxy+ty^2,\quad (h>0), \text{ ....}(1)$$

and the negative radii of curvature are proportional to the squares of the semidiameters of the indicatrix

$$z=-h,\quad -2h=rx^2+2sxy+ty^2,\quad (h>0). \text{ ....}(2)$$

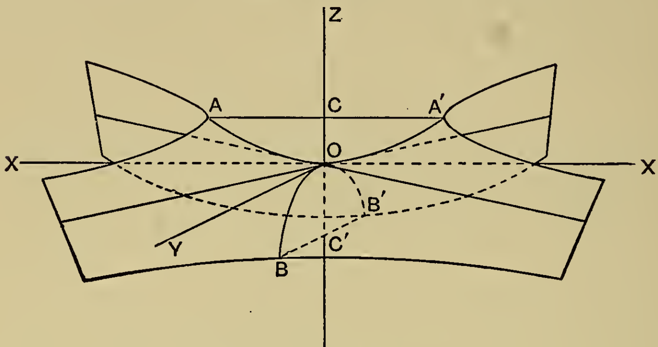

FIG. 61.

The normal section of algebraically greatest curvature passes through the real axis of the indicatrix (1), and the normal section of algebraically least curvature through the real axis of the indicatrix (2). These indicatrices project on the xy-plane into conjugate hyperbolas whose common asymptotes are the inflexional tangents. As in § 220, the sections of greatest and least curvature are the principal sections. If the axes OX and OY lie in the principal sections the equations to the indicatrices are

$$(1)\ z=h,\quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=1;\quad (2)\ z=-h,\quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=-1.$$

If ρ_1, ρ_2 measure the principal radii in magnitude and sign,

$$\rho_1=\text{Lt}\,\frac{a^2}{2h},\quad \rho_2=\text{Lt}\left(\frac{-b^2}{2h}\right),$$

and therefore the equations to the indicatrices are

$$(1)\ z=h,\quad \frac{x^2}{\rho_1}+\frac{y^2}{\rho_2}=2h,\quad (2)\ z=-h,\quad \frac{x^2}{\rho_1}+\frac{y^2}{\rho_2}=-2h,$$

and the equation to the surface is

$$2z=\frac{x^2}{\rho_1}+\frac{y^2}{\rho_2}+\dots.$$

The radius of curvature of the normal section that makes an angle θ with the zx-plane is given by

$$\frac{1}{\rho}=\frac{\cos^2\theta}{\rho_1}+\frac{\sin^2\theta}{\rho_2}.$$

222. Curvature of normal sections through a parabolic point. If $rt-s^2=0$, the indicatrix consists of two parallel straight lines, (fig. 62). The inflexional tangents

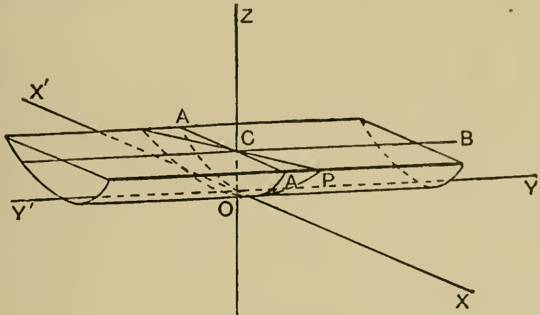

FIG. 62.

coincide, and the normal section which contains them has its curvature zero. The normal section at right angles to the section of zero curvature has maximum curvature. These two sections are the principal sections. If OX and OY lie in the principal sections, the equations to the indicatrix are

$$z=h,\quad x^2=a^2,$$

where $a=$ CA. The finite principal radius ρ_1 is given by

$$\rho_1=\text{Lt}\,\frac{a^2}{2h}.$$

Hence the equations to the indicatrix and surface are

$$z=h,\quad 2h=\frac{x^2}{\rho_1};\quad 2z=\frac{x^2}{\rho_1}+\dots.$$

If ρ is the radius of curvature of the section OCP which makes an angle θ with the principal section OCA,

$$\frac{1}{\rho}=\frac{\cos^2\theta}{\rho_1}.$$

223. Umbilics. If $r=t$ and $s=0$, the indicatrix is a circle and the principal sections are indeterminate, since all normal sections have the same curvature. Points at which the indicatrix is circular are **umbilics.**

224. The curvature of an oblique section. The relation between the curvatures of a normal section and an oblique section through the same tangent line is stated in **Meunier's Theorem**: *If ρ_0 and ρ are the radii of curvature of a normal section and an oblique section through the same tangent, $\rho=\rho_0 \cos\theta$, where θ is the angle between the sections.*

If the tangent plane at the point is taken as xy-plane, the normal as z-axis, and the common tangent to the sections as x-axis, the equations to the indicatrix are

$$z=h, \quad 2h=rx^2+2sxy+ty^2,$$

and, (see fig. 60),

$$\rho_0=\mathrm{Lt}\frac{\mathrm{CA}^2}{2\mathrm{OC}}=\frac{1}{r}.$$

The equations to QQ′ are

$$y=h\tan\theta, \quad z=h,$$

and where QQ′ meets the surface,

$$2h=rx^2+2sxh\tan\theta+th^2\tan^2\theta.$$

But if x and y are small quantities of the first order, h is of the second order, and therefore to our degree of approximation, hx and h^2 may be rejected. Hence $\mathrm{QV}^2=\dfrac{2h}{r}$, and

$$\rho=\mathrm{Lt}\frac{\mathrm{QV}^2}{2\mathrm{OV}}=\mathrm{Lt}\frac{2h/r}{2h\sec\theta}=\rho_0\cos\theta.$$

The following proof of Meunier's theorem is due to Besant.

Let OT be the common tangent to the sections and consider the sphere which touches OT at O and passes through an adjacent and ultimately coincident point on each section. The planes of the sections cut the sphere in circles which are the circles of curvature at O of the sections. The circle in the plane containing the normal is a great circle, and the other is a small circle of the sphere. If C_0 is the centre of the great circle and C of the small circle, the triangle COC_0 is right angled at C, and the angle COC_0 is the angle between the planes. Hence the theorem immediately follows.

225. Expression for radius of curvature of a given section through any point of a surface. Let OT, the tangent to a given section of a surface through a given point O on it, have direction-cosines l_1, m_1, n_1. Let the normal to OT which lies in the plane of the section have direction-cosines l_2, m_2, n_2. Then, since the direction-cosines of the normal to the surface are

$$\frac{-p}{\sqrt{1+p^2+q^2}}, \quad \frac{-q}{\sqrt{1+p^2+q^2}}, \quad \frac{1}{\sqrt{1+p^2+q^2}},$$

θ, the angle between the plane of the section and the normal section through OT, is given by

$$\cos\theta = \frac{-pl_2 - qm_2 + n_2}{\sqrt{1+p^2+q^2}}.$$

But $$pl_1 + qm_1 - n_1 = 0;$$

therefore, by Frenet's formulae, since

$$\frac{dp}{ds} = \frac{\partial p}{\partial x}\cdot\frac{dx}{ds} + \frac{\partial p}{\partial y}\cdot\frac{dy}{ds} = rl_1 + sm_1$$

and $$\frac{dq}{ds} = \frac{\partial q}{\partial x}\cdot\frac{dx}{ds} + \frac{\partial q}{\partial y}\cdot\frac{dy}{ds} = sl_1 + tm_1,$$

$$\frac{pl_2 + qm_2 - n_2}{\rho} = -(rl_1^2 + 2sl_1m_1 + tm_1^2)$$

or $$\frac{\cos\theta}{\rho} = \frac{rl_1^2 + 2sl_1m_1 + tm_1^2}{\sqrt{1+p^2+q^2}}.$$

Cor. When $\theta = 0$, ρ becomes ρ_0, and Meunier's theorem immediately follows.

Ex. 1. Find the principal radii at the origin of the paraboloid

$$2z = 5x^2 + 4xy + 2y^2.$$

Find also the radius of curvature of the section $x = y$. *Ans.* 1, $\frac{1}{6}$; $\frac{2}{11}$.

Ex. 2. For the hyperbolic paraboloid

$$2z = 7x^2 + 6xy - y^2,$$

prove that the principal radii at the origin are $\frac{1}{8}$ and $-\frac{1}{2}$, and that the principal sections are

$$x = 3y, \quad 3x = -y.$$

Ex. 3. If ρ, ρ' are the radii of curvature of any two perpendicular normal sections at a point of a surface, $\frac{1}{\rho} + \frac{1}{\rho'}$ is constant.

Ex. 4. Prove that at the origin the radius of curvature of the section of the surface
$$ax^2+2hxy+by^2=2z,$$
by the plane
$$lx+my+nz=0,$$
is
$$(l^2+m^2)^{\frac{3}{2}}(am^2-2hlm+bl^2)^{-1}(l^2+m^2+n^2)^{-\frac{1}{2}}.$$

Ex. 5. The locus of the centres of curvature of sections of the surface
$$2z=\frac{x^2}{\rho_1}+\frac{y^2}{\rho_2}+\ldots$$
which pass through the origin is the surface given by
$$(x^2+y^2+z^2)\left(\frac{x^2}{\rho_2}+\frac{y^2}{\rho_1}\right)=z(x^2+y^2).$$

226. Principal radii at a point of an ellipsoid. Let P be a point on an ellipsoid, centre O. Take OP as z-axis and the diametral plane of OP as xy-plane. Then take the principal axes of the section of the ellipsoid by the xy-plane as x- and y-axes. Since the coordinate axes are conjugate diameters of the ellipsoid, its equation is
$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1,$$
where $\gamma=$OP, and 2α and 2β are the principal axes of the section of the ellipsoid by the plane through the centre which is parallel to the tangent plane at P.

The equations to the indicatrix are $z=\gamma-k$, where k is small, and
$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1-\left(\frac{\gamma-k}{\gamma}\right)^2=\frac{2k}{\gamma}.$$

Therefore, if the axes of the indicatrix are a and b,
$$a^2=\frac{2k\alpha^2}{\gamma},\quad b^2=\frac{2k\beta^2}{\gamma}.$$

Now let p be the perpendicular from the centre to the tangent plane at P, and let h be the distance between the planes of the indicatrix and the tangent plane. Then
$$\frac{k}{\gamma}=\frac{h}{p}.$$

Therefore, if the principal radii are ρ_1 and ρ_2,
$$\rho_1=\text{Lt}\frac{a^2}{2h}=\frac{\alpha^2}{p}\quad\text{and}\quad\rho_2=\text{Lt}\frac{b^2}{2h}=\frac{\beta^2}{p}.$$

Ex. 1. Prove that the principal radii at a point (x, y, z) on the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ are given by

$$\rho^2-\frac{\rho}{p}\left(x^2\frac{b^2+c^2}{a^2}+y^2\frac{c^2+a^2}{b^2}+z^2\frac{a^2+b^2}{c^2}\right)+\frac{a^2b^2c^2}{p^4}=0. \quad \text{(Use § 86.)}$$

Ex. 2. If PT is tangent to a normal section at P on an ellipsoid, the radius of curvature of the section is r^2/p, where r is the central radius parallel to PT.

Ex. 3. If λ, μ are the parameters of the confocals through a point P on the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, the principal radii at P are

$$\frac{\sqrt{\lambda^3\mu}}{abc}, \quad \frac{\sqrt{\lambda\mu^3}}{abc}.$$

Ex. 4. The normal at a point P of an ellipsoid meets the principal planes through the mean axis in Q and R. If the sum of the principal radii at P equal to PQ+PR, prove that P lies on a real central circular section of the ellipsoid.

Ex. 5. If P is an umbilic of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, prove that the curvature at P of any normal section through P is ac/b^3. (See § 95, Ex. 2.)

LINES OF CURVATURE.

227. A curve drawn on a surface so that its tangent at any point touches one of the principal sections of the surface at the point is called a **line of curvature**. There pass, in general, two lines of curvature through every point of the surface, and the two lines of curvature through any point cut at right angles.

228. Lines of curvature of an ellipsoid. The tangents to the principal sections at a point P of an ellipsoid whose centre is O are parallel to the axes of the central section of the ellipsoid by the diametral plane of OP, (§ 226). But the tangents to the curves of intersection of the ellipsoid and the confocal hyperboloids through P are also parallel to the axes of the section, (§ 121). Therefore the lines of curvature on the ellipsoid are its curves of intersection with confocal hyperboloids.

229. Lines of curvature on a developable surface. One principal section at any point of a developable is the normal section through the generator. Hence the generators form one system of lines of curvature. The

other system consists of curves drawn on the surface to cut all the generators at right angles. In the case of a cone, the curve of intersection of the cone and any sphere which has its centre at the vertex cuts all the generators at right angles, and therefore the second system of lines of curvature consists of the curves of intersection of the cone and concentric spheres whose centres are at the vertex.

230. The normals to a surface at points of a line of curvature. A fundamental property of lines of curvature may be stated as follows:

If O *and* P *are adjacent and ultimately coincident points of a line of curvature, the normals to the surface at* O *and* P *intersect; conversely, if* O *and* P *are adjacent points of a curve drawn on a surface and the normals to the surface at* O *and* P *intersect, the curve is a line of curvature of the surface.*

Let O be the origin and let the equation to the surface be

$$2z=\frac{x^2}{\rho_1}+\frac{y^2}{\rho_2}+\dots.$$

P will lie on the indicatrix and will have coordinates $r\cos\theta$, $r\sin\theta$, h. The equations to the normal at P to the surface are

$$\frac{x-r\cos\theta}{\dfrac{r\cos\theta}{\rho_1}}=\frac{y-r\sin\theta}{\dfrac{r\sin\theta}{\rho_2}}=\frac{z-h}{-1}.$$

Therefore, if the normal at P is coplanar with the normal at O, *i.e.* with OZ,

$$\sin\theta\cos\theta\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)=0. \quad\text{.................}(1)$$

If O and P are adjacent points of a line of curvature, $\sin\theta=0$, or $\cos\theta=0$, and the condition (1) is satisfied; therefore the normals at adjacent points of a line of curvature intersect.

If the normals at O and P intersect, $\cos\theta=0$ or $\sin\theta=0$, and therefore O and P are adjacent points of one of the principal sections, or the curve is a line of curvature.

Ex. The normals to an ellipsoid at its points of intersection with a confocal generate a developable surface.

231. Lines of curvature on a surface of revolution. The normals to a surface of revolution at all points of a meridian section lie in the plane of the section, and therefore, by § 230, the meridian sections are lines of curvature. The normals at all points of a circular section pass through the same point on the axis, and therefore any circular section is a line of curvature.

Let P, (fig. 63), be any point on the surface, and let PT and PK be the tangents to the meridian and circular sections through P. Let PN be the normal at P, meeting the axis in N, and let C be the centre of the circular section. Then

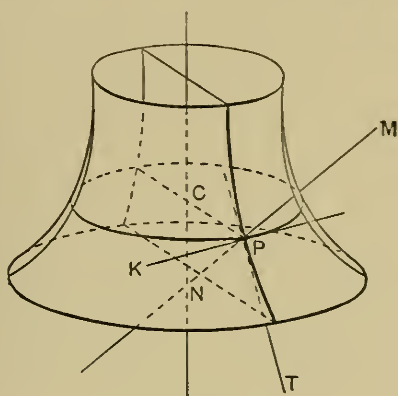

FIG. 63.

TPN and KPN are the planes of the principal sections. The principal radius in the plane TPN is the radius of curvature at P of the generating curve. The circular section is an oblique section through the tangent PK, and its radius of curvature is CP. Therefore, by Meunier's theorem, if ρ is the principal radius in the plane KPN,

$$\text{CP} = \rho \cos\theta, \text{ where } \theta = \angle \text{CPN},$$

or

$$\rho = \text{PN}.$$

Thus the other principal radius is the intercept on the normal between P and the axis.

Ex. 1. In the surface formed by the revolution of a parabola about its directrix one principal radius at any point is twice the other.

Ex. 2. For the surface formed by the revolution of a catenary about its directrix, (the **catenoid**), the principal radii at any point are equal and of opposite sign.

(A surface whose principal radii at each point are equal and of opposite sign is called a **minimal surface**.)

Ex. 3. In the conicoid formed by the revolution of a central conic about an axis one principal radius varies as the cube of the other.

Ex. 4. A developable surface is generated by the tangents to a given curve. Prove that at the point Q on the tangent at P, where $PQ=l$, the principal radius of the developable is $\dfrac{l\sigma}{\rho}$.

Let the plane through Q at right angles to PQ cut the consecutive generators in N and M, (fig. 64). Then N, Q, M are consecutive points

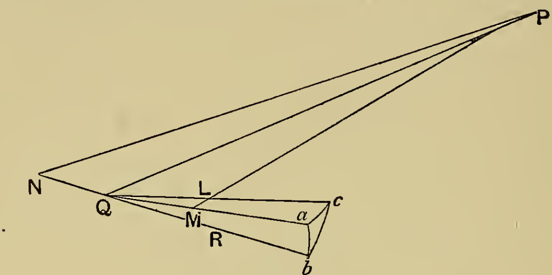

FIG. 64.

of the principal section. But the angle between consecutive generators is $\delta\psi$, and the angle between the planes PQN, PQM is the angle between consecutive osculating planes, and therefore is $\delta\tau$. Therefore, if ρ_1 is the principal radius,

$$\rho_1=\text{Lt}\frac{QM}{\angle RQM}=\text{Lt}\frac{l\delta\psi}{\delta\tau}=\frac{l\sigma}{\rho}.$$

Ex. 5. Find the radius of curvature at Q of the line of curvature of the developable.

Draw QL at right angles to the consecutive generator. Then N, Q, L are consecutive points on the line of curvature. Let QM, NQ, QL meet the sphere of unit radius whose centre is Q in a, b, c respectively. Then, if $\delta\psi'$ is the angle between consecutive tangents to the line of curvature,

$$\delta\psi'=bc,\quad \delta\psi=ca,\quad \delta\tau=ab.$$

Therefore, since the triangle cab is right-angled at a,

$$\delta\psi'^2=\delta\psi^2+\delta\tau^2.$$

If ρ_0 is the radius of curvature of the line of curvature, we have

$$\rho_0=\text{Lt}\frac{QL}{\delta\psi'}=\text{Lt}\frac{l\delta\psi}{\delta\psi'}.$$

Hence
$$\frac{1}{\rho_0^2}=\text{Lt}\frac{1}{l^2}\left\{1+\left(\frac{\delta\tau}{\delta\psi}\right)^2\right\},$$

and
$$\rho_0=\frac{l\sigma}{\sqrt{\rho^2+\sigma^2}}.$$

Ex. 6. Shew that the radius of torsion of the line of curvature of the developable is

$$\frac{l(\rho^2+\sigma^2)}{\rho(\sigma\rho'-\rho\sigma')},$$

where $$\rho' \equiv \frac{d\rho}{ds}, \qquad \sigma' \equiv \frac{d\sigma}{ds},$$

and s is the arc of the edge of regression.

Ex. 7. Shew that the lines of curvature of the developable generated by tangents to a helix are plane curves.

232. The principal radii and lines of curvature through a point of the surface $z=f(x, y)$. In § 225 we have found that if l_1, m_1, n_1 are the direction-cosines of the tangent to a normal section of the surface through the point (x, y, z), the radius of curvature of the section is given by

$$\frac{1}{\rho}=\frac{rl_1^2+2sl_1m_1+tm_1^2}{\sqrt{1+p^2+q^2}}. \quad \text{..................(1)}$$

We have also $$pl_1+qm_1-n_1=0,$$

whence $$l_1^2+m_1^2+(pl_1+qm_1)^2=1. \text{..................(2)}$$

Therefore, if we write k for $\sqrt{1+p^2+q^2}$, we may combine (1) and (2) into

$$l_1^2\left(1+p^2-\frac{r\rho}{k}\right)+2l_1m_1\left(pq-\frac{s\rho}{k}\right)+m_1^2\left(1+q^2-\frac{t\rho}{k}\right)=0. \text{..(3)}$$

This equation gives two values of $l_1:m_1$, which correspond to the two sections through the point which have a given radius of curvature. If ρ is a principal radius, these sections coincide, (cf. § 85). Therefore the principal radii are given by

$$\left(1+p^2-\frac{r\rho}{k}\right)\left(1+q^2-\frac{t\rho}{k}\right)=\left(pq-\frac{s\rho}{k}\right)^2, \text{.........(4)}$$

or $$\rho^2(rt-s^2)-k\rho\{(1+p^2)t+(1+q^2)r-2spq\}+k^4=0.$$

If equation (4) is satisfied, the coincident values of $l_1:m_1$ are

$$-\frac{l_1}{m_1}=\frac{pq-\frac{s\rho}{k}}{1+p^2-\frac{r\rho}{k}}=\frac{1+q^2-\frac{t\rho}{k}}{pq-\frac{s\rho}{k}}. \quad \text{............(5)}$$

Now if PQ is a straight line whose direction-cosines are l_1, m_1, n_1, and P′Q′ is its projection on the xy-plane, the projections of PQ and P′Q′ on the x- and y-axes are identical, and therefore the gradient of P′Q′ with reference to the axes OX and OY is m_1/l_1. Hence, from (5), the differential equation to the projection on the xy-plane of the line of curvature corresponding to the radius ρ is

$$\frac{dy}{dx}+\frac{1+p^2-\frac{r\rho}{k}}{pq-\frac{s\rho}{k}}=0 \quad \text{or} \quad \frac{dy}{dx}+\frac{pq-\frac{s\rho}{k}}{1+q^2-\frac{t\rho}{k}}=0,$$

which may be written

$$dx\left(1+p^2-\frac{r\rho}{k}\right)+dy\left(pq-\frac{s\rho}{k}\right)=0$$

or

$$dx\left(pq-\frac{s\rho}{k}\right)+dy\left(1+q^2-\frac{t\rho}{k}\right)=0.$$

If we eliminate ρ/k between these equations, we obtain the differential equation to the projections of the two lines of curvature, viz.,

$$dx^2\{s(1+p^2)-rpq\}+dx\,dy\{t(1+p^2)-r(1+q^2)\}$$
$$+dy^2\{tpq-s(1+q^2)\}=0.$$

Ex. 1. Shew that the principal radii at a point of the paraboloid $\frac{x^2}{a}+\frac{y^2}{b}=2z$ are given by

$$\rho^2-k\rho(a+b+2z)+abk^4=0,$$

where
$$k^2\equiv 1+\frac{x^2}{a^2}+\frac{y^2}{b^2}.$$

Ex. 2. Prove that at a point of the intersection of the paraboloid $\frac{x^2}{a}+\frac{y^2}{b}=2z$ and the confocal $\frac{x^2}{a-\lambda}+\frac{y^2}{b-\lambda}=2z-\lambda$, the principal radii are $k\lambda$, $\frac{abk^3}{\lambda}$, where $k^2\equiv\frac{\lambda(a+b+2z)-\lambda^2}{ab}$.

Ex. 3. Shew that the projections on the xy-plane of the lines of curvature of the paraboloid $\frac{x^2}{a}+\frac{y^2}{b}=2z$ are given by

$$xy\left\{a\left(\frac{dy}{dx}\right)^2-b\right\}+\frac{dy}{dx}\{b(a^2+x^2)-a(b^2+y^2)\}=0.$$

Ex. 4. Prove that when $a=b$ this equation reduces to

$$\frac{dy}{dx}=\frac{y}{x} \quad \text{or} \quad \frac{dy}{dx}=-\frac{x}{y},$$

whence $y=Ax$ or $x^2+y^2=B$, where A and B are arbitrary constants. Shew that this verifies the results of § 231 for the paraboloid.

Ex. 5. Prove that the integral of the equation in Ex. 3 is

$$y^2-\lambda x^2=\frac{ab(a-b)\lambda}{b+a\lambda},$$

where λ is an arbitrary constant, and shew that this becomes

$$\frac{x^2}{a(a-\mu)}+\frac{y^2}{b(b-\mu)}+1=0,$$

if

$$\mu\equiv\frac{a^2\lambda+b^2}{a\lambda+b}.$$

Hence prove that the lines of curvature of the paraboloid are its curves of intersection with confocals.

Ex. 6. Prove that the intersection of the surface $3z=ax^3+by^3$ and the plane $ax=by$ is a line of curvature of the surface.

Ex. 7. Prove that the condition that the normal,

$$\frac{\xi-x}{p}=\frac{\eta-y}{q}=\frac{\zeta-z}{-1},$$

to the surface $z=f(x, y)$ at a point of a curve drawn on it should intersect the consecutive normal is

$$\frac{dp}{dx+p\,dz}=\frac{dq}{dy+q\,dz},$$

and deduce the equation to the lines of curvature obtained in § 232.
Apply § 48. Also $dz=pdx+qdy$, $dp=rdx+sdy$, $dq=sdx+tdy$.

Ex. 8. If l_1, m_1, n_1 are the direction-cosines of the tangent to a line of curvature, and l, m, n are the direction-cosines of the normal to the surface at the point,

$$\frac{dl}{l_1}=\frac{dm}{m_1}=\frac{dn}{n_1}.$$

Ex. 9. Prove that at a point of a line of curvature of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$,

$$\frac{x}{dx}(b^2-c^2)+\frac{y}{dy}(c^2-a^2)+\frac{z}{dz}(a^2-b^2)=0,$$

and shew that the coordinates of any point of the curve of intersection of the ellipsoid and the confocal $\dfrac{x^2}{a^2+\lambda}+\dfrac{y^2}{b^2+\lambda}+\dfrac{z^2}{c^2+\lambda}=1$ verify this equation.

Ex. 10. Prove that for the helicoid $z=c\tan^{-1}\dfrac{y}{x}$,

$$\rho_1=-\rho_2=\frac{u^2+c^2}{c}, \text{ where } u^2\equiv x^2+y^2.$$

Any point, P, on the surface is given by

$$x=u\cos\theta,\quad y=u\sin\theta,\quad z=c\theta.$$

The tangent plane at P is

$$x\sin\theta-y\cos\theta+\frac{uz}{c}-u\theta=0,$$

and hence, if p is the perpendicular to the plane from the point Q, "$u+\delta u,\ \theta+\delta\theta$,"

$$p=\frac{c\,\delta u\,\delta\theta}{\sqrt{u^2+c^2}}.$$

But if d is the distance PQ, the radius of curvature at P of the normal section through Q is given by

$$\rho=\text{Lt}\frac{d^2}{2p}=\text{Lt}\frac{\{(u^2+c^2)\,\delta\theta^2+\delta u^2\}\sqrt{u^2+c^2}}{2c\,\delta u\,\delta\theta}.$$

Therefore $$(u^2+c^2)\,d\theta^2-2c\frac{du\,d\theta}{\sqrt{u^2+c^2}}\rho+du^2=0.$$

This gives two values of $d\theta : du$, which correspond to the two sections with radius ρ. If ρ is a principal radius, as in § 232, we have coincident values of $d\theta : du$.

Hence $$\rho=\pm\frac{u^2+c^2}{c}.$$

The differential equation to the projections on the xy-plane of the lines of curvature is

$$d\theta=\pm\frac{du}{\sqrt{u^2+c^2}},$$

where u and θ are polar coordinates. Hence the lines of curvature are the intersections of the helicoid and cylinders

$$2u=c(Ae^{\theta}-A^{-1}e^{-\theta}),$$

where A is an arbitrary constant.

Ex. 11. For the helicoid $z=c\tan^{-1}\dfrac{y}{x}$, prove that

$$p=\frac{-cy}{u^2},\quad q=\frac{cx}{u^2},\quad r=-t=\frac{2cxy}{u^4},\quad s=\frac{c(y^2-x^2)}{u^4},$$

and deduce the results of Ex. 10 from the equations of § 232.

Ex. 12. Prove that at a point of the conoid

$$x=u\cos\theta,\quad y=u\sin\theta,\quad z=f(\theta),$$

the principal radii are given by

$$z'^2\rho^2-u^2kz''\rho-u^2k^2(u^2+z'^2)=0,$$

where $$z'\equiv\frac{dz}{d\theta},\ \text{etc., and}\ k^2\equiv1+\frac{z'^2}{u^2}.$$

We have $$p=-\frac{\sin\theta}{u}z',\quad q=\frac{\cos\theta}{u}z',\quad r=\frac{z''\sin^2\theta+z'\sin2\theta}{u^2},$$

$$s=-\frac{z''\sin\theta\cos\theta+z'\cos2\theta}{u^2},\quad t=\frac{z''\cos^2\theta-z'\sin2\theta}{u^2}.$$

Ex. 13. Prove that at a point of the surface of revolution

$$x = u\cos\theta, \quad y = u\sin\theta, \quad z = f(u),$$

the principal radii are

$$\rho_1 = \frac{-u\sqrt{1+z'^2}}{z'}, \quad \rho_2 = \frac{-(1+z'^2)^{\frac{3}{2}}}{z''},$$

where $z' \equiv \dfrac{dz}{du}$, etc. Deduce the result of § 231.

Ex. 14. For the surface

$$x = u\cos\theta, \quad y = u\sin\theta, \quad z = c\log(u+\sqrt{u^2-c^2}),$$

prove that $\rho_1 = -\rho_2$.

Ex. 15. Find the principal radii at a point of the cylindroid $z(x^2+y^2)=2mxy$. Prove that at any point of the generator $x=y$, $z=m$, one principal radius is infinite and the other is $\dfrac{u^2}{4m}$, where u is the distance of the point from the z-axis.

Ex. 16. Find the curvature at the origin of the lines of curvature of the surface

$$2z = \frac{x^2}{\rho_1} + \frac{y^2}{\rho_2} + \frac{1}{3}(ax^3+3bx^2y+3cxy^2+dy^3)+\dots.$$

If l, m, n are the direction-cosines of the tangent to a curve and α is the arc,

$$\frac{1}{\rho^2} = \left(\frac{dl}{d\alpha}\right)^2 + \left(\frac{dm}{d\alpha}\right)^2 + \left(\frac{dn}{d\alpha}\right)^2.$$

But for a line of curvature,

$$l^2\{rpq-s(1+p^2)\}+lm\{r(1+q^2)-t(1+p^2)\}+m^2\{s(1+q^2)-tpq\}=0, \quad \dots(1)$$

and for the line of curvature that touches OX, $l=1$, $m=n=0$. Also at the origin $p=q=s=0$; therefore differentiating (1), and substituting, we obtain

$$\frac{dm}{d\alpha} = \frac{\dfrac{ds}{d\alpha}}{r-t} = \frac{1}{r-t}\left(l\frac{\partial s}{\partial x}+m\frac{\partial s}{\partial y}\right) = \frac{bl}{r-t} = \frac{b}{\dfrac{1}{\rho_1}-\dfrac{1}{\rho_2}}.$$

Again, $pl+qm=n$, and therefore at the origin

$$\frac{dn}{d\alpha} = rl^2+tm^2 = r = \frac{1}{\rho_1}.$$

And, since $\qquad l\dfrac{dl}{d\alpha}+m\dfrac{dm}{d\alpha}+n\dfrac{dn}{d\alpha}=0, \quad \dfrac{dl}{d\alpha}=0.$

Therefore, if ρ_0 is the radius of curvature of the line of curvature which touches OX,

$$\frac{1}{\rho_0^2} = \frac{1}{\rho_1^2} + \frac{b^2}{(1/\rho_1-1/\rho_2)^2}.$$

Similarly, the square of the curvature of the line of curvature that touches OY is

$$\frac{1}{\rho_2^2} + \frac{c^2}{(1/\rho_1-1/\rho_2)^2}.$$

Ex. 17. Prove that the equation to an ellipsoid can be put in the form

$$2z=\frac{x^2}{\rho_1}+\frac{y^2}{\rho_2}-\frac{1}{p}\left(\frac{x^3\xi}{\rho_1{}^2}+\frac{x^2y\eta}{\rho_1\rho_2}+\frac{xy^2\xi}{\rho_1\rho_2}+\frac{y^3\eta}{\rho_2{}^2}\right)+\ldots,$$

where ξ, η, p are the coordinates of the centre.

Ex. 18. Hence shew that the squares of the curvatures of the lines of curvature through a point P are

$$\frac{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}{\lambda(\lambda-\mu)^3}+\frac{a^2b^2c^2}{\lambda^3\mu},\quad \frac{(a^2-\mu)(b^2-\mu)(c^2-\mu)}{\mu(\mu-\lambda)^3}+\frac{a^2b^2c^2}{\mu^3\lambda},$$

where λ and μ are the parameters of the confocals through P.

Ex. 19. PN, $\mathrm{PN_1}$, $\mathrm{PN_2}$ are the normals at a point P to an ellipsoid and the confocal hyperboloids of one and two sheets through P. Prove that the curvature at P of the curve of section of the ellipsoid and hyperboloid of two sheets is $(\rho_1{}^{-2}+\rho_2{}^{-2})^{\frac{1}{2}}$, where $\rho_1{}^{-1}$ is the curvature of the section of the ellipsoid by the plane $\mathrm{PNN_1}$, and $\rho_2{}^{-1}$ is the curvature of the section of the hyperboloid of two sheets by the plane $\mathrm{PN_1N_2}$.

233. Umbilics. At an umbilic the directions of the principal sections are indeterminate; therefore, from equations (5) of § 232, we have

$$\frac{1+p^2}{r}=\frac{1+q^2}{t}=\frac{pq}{s}=\frac{\rho}{k},$$

where ρ is the radius of curvature of any normal section through the umbilic.

Ex. 1. Find the umbilics of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$. By differentiation, we obtain

$$\frac{x}{a^2}+\frac{pz}{c^2}=0,\quad \text{or}\quad p=\frac{-c^2x}{a^2z},$$

$$\frac{y}{b^2}+\frac{qz}{c^2}=0,\quad \text{or}\quad q=\frac{-c^2y}{b^2z}.$$

Whence $\quad r=\dfrac{-1}{z}\left(\dfrac{c^2}{a^2}+p^2\right),\quad s=\dfrac{-pq}{z},\quad t=\dfrac{-1}{z}\left(\dfrac{c^2}{b^2}+q^2\right).$

At an umbilic $s(1+p^2)=rpq$ or $pq(1+p^2)+rzpq=0$.

Hence $p=0$ or $q=0$; $(rz+1+p^2\neq 0$ unless $c^2+a^2=0)$.

We have also at the umbilic

$$t(1+p^2)-r(1+q^2)=0$$

or

$$a^2p^2(b^2-c^2)+b^2q^2(c^2-a^2)=c^2(a^2-b^2).$$

If $a>b>c$, $p=0$ gives imaginary values of q.

If $$q=0,\quad p=\pm\frac{c}{a}\sqrt{\frac{a^2-b^2}{b^2-c^2}}.$$

Therefore, since $$p=\frac{-c^2x}{a^2z},$$

$$\frac{x/a}{\sqrt{a^2-b^2}}=\frac{y/b}{0}=\frac{z/c}{\pm\sqrt{b^2-c^2}}=\frac{1}{\pm\sqrt{a^2-c^2}}. \quad \text{(Cf. § 95.)}$$

At an umbilic $$k^2=1+p^2+q^2=\frac{b^2(a^2-c^2)}{a^2(b^2-c^2)}.$$

Hence, $$\rho=\frac{k}{t}, \quad \text{since } q=0,$$

$$=\frac{-kb^2z}{c^2}=-\frac{b^3}{ac}.$$

(Cf. § 226, Ex. 5.)

Ex. 2. Shew that the points of intersection of the surface

$$x^m+y^m+z^m=a^m$$

and the line $x=y=z$ are umbilics, and that the radius of curvature at an umbilic is given by

$$\rho=\frac{a}{m-1}(3)^{\frac{m-2}{2m}}.$$

Ex. 3. Prove that the surface

$$a^3x^3+b^3y^3+c^3z^3=(x^2+y^2+z^2)^3$$

has an umbilic where it meets the line

$$a^3x=b^3y=c^3z.$$

Ex. 4. Prove that in general three lines of curvature pass through an umbilic.

If the umbilic is taken as origin, the equation to the surface is

$$2z=\frac{x^2+y^2}{\rho}+\frac{1}{3}(ax^3+3bx^2y+3cxy^2+dy^3)+\dots.$$

The condition that the normal at (x, y, z) should intersect the normal at O is $$bx^3+x^2y(2c-a)+xy^2(d-2b)-cy^3=0.$$

Therefore, if the tangent to a line of curvature makes an angle α with the x-axis,

$$\tan\alpha=\text{Lt}\frac{y}{x}$$

and $$c\tan^3\alpha+(2b-d)\tan^2\alpha-(2c-a)\tan\alpha-b=0.$$

This equation gives three values of $\tan\alpha$ which correspond to the three lines of curvature through the umbilic.

Ex. 5. If the origin is an umbilic of the surface $z=f(x, y)$, the directions of the three lines of curvature through the origin are given by

$$\frac{\partial t}{\partial x}\tan^3\alpha+\left(2\frac{\partial r}{\partial y}-\frac{\partial t}{\partial y}\right)\tan^2\alpha-\left(2\frac{\partial t}{\partial x}-\frac{\partial r}{\partial x}\right)\tan\alpha-\frac{\partial r}{\partial y}=0.$$

Ex. 6. Investigate the lines of curvature through an umbilic of an ellipsoid.

If the umbilic is the origin, the normal at the origin the z-axis, and the principal plane which contains the umbilics the zx-plane, the

equation to the ellipsoid is

$$x^2+y^2+cz^2+2gzx+2wz=0.$$

Whence, at the origin, we have

$$\frac{\partial t}{\partial x}=\frac{g}{w^2},\quad \frac{\partial r}{\partial x}=\frac{3g}{w^2},\quad \frac{\partial r}{\partial y}=\frac{\partial t}{\partial y}=0.$$

Therefore the directions of the lines of curvature through the origin are given by

$$\tan^3\alpha+\tan\alpha=0,$$

and the only real line of curvature through the umbilics is the section of the ellipsoid by the principal plane that contains the umbilics.

Ex. 7. Shew that the points of intersection of the line $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$ and surface $\frac{x^3}{a}+\frac{y^3}{b}+\frac{z^3}{c}=k^2$ are umbilics on the surface, and that the directions of the three lines of curvature through an umbilic (x, y, z) are given by

$$\frac{dy}{b}=\frac{dz}{c},\quad \frac{dz}{c}=\frac{dx}{a},\quad \frac{dx}{a}=\frac{dy}{b}.$$

If P and Q are adjacent and ultimately coincident points of a curve drawn on the surface, the normals at P and Q intersect if

$$2\Sigma z^2 dx\,dy\left(\frac{x}{a}-\frac{y}{b}\right)+\Sigma z^2 dx\,dy\left(\frac{dx}{a}-\frac{dy}{b}\right)=0.$$

Also, we have

$$\frac{x^2dx}{a}+\frac{y^2dy}{b}+\frac{z^2dz}{c}=0,$$

since the tangent to the curve lies in the tangent plane to the surface. If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, these equations give three values of $dx:dy:dz$, and the first equation then reduces to

$$(c\,dy-b\,dz)(a\,dz-c\,dx)(b\,dx-a\,dy)=0.$$

234. Triply-orthogonal systems of surfaces. When three systems of surfaces are such that through each point in space there passes one member of each system, and the three members through any given point cut at right angles, they are together said to form a **triply-orthogonal system** of surfaces. The confocals of a given conicoid form such a system.

We have seen that the lines of curvature of an ellipsoid are its curves of intersection with the confocal hyperboloids. This is a particular case of a general theorem on the lines of curvature of a triply-orthogonal system, **Dupin's theorem**, which we proceed to enunciate and prove.

If three systems of surfaces cut everywhere at right angles, the lines of curvature of any member of one system

are its curves of intersection with the members of the other two systems.

Let O be any point on a given surface, S_1, of the first system, and let S_2 and S_3 be the surfaces of the second and third systems that pass through O. We have to prove that the curves of intersection of S_1 with S_2 and S_3 are the lines of curvature on S_1. The tangent planes at O to the three surfaces cut at right angles. Take them for coordinate planes. The equations to the three surfaces are then,

to S_1, $\qquad 2x + a_1y^2 + 2h_1yz + b_1z^2 + \ldots = 0,$

to S_2, $\qquad 2y + a_2z^2 + 2h_2zx + b_2x^2 + \ldots = 0,$

to S_3, $\qquad 2z + a_3x^2 + 2h_3xy + b_3y^2 + \ldots = 0.$

Near the origin, on the curve of intersection of the surfaces S_1 and S_2, x and y are of the second order of small quantities, and hence the coordinates of a point of the curve adjacent to O are $0, 0, \gamma$. The tangent planes to S_1 and S_2 at $(0, 0, \gamma)$ are, if γ^2 be rejected, given by

$$x + h_1\gamma y + b_1\gamma z = 0,$$

$$y + a_2\gamma z + h_2\gamma x = 0,$$

and they are at right angles;

therefore $\qquad h_1 + h_2 = 0.$

Similarly, we have

$$h_2 + h_3 = 0, \quad h_3 + h_1 = 0,$$

and therefore $\qquad h_1 = h_2 = h_3 = 0.$

Hence the coordinate planes are the planes of the principal sections at O of the three surfaces and the curve of intersection of S_1 and S_2 touches a principal section of S_1 at O. But O is any point of S_1, and therefore the curve touches a principal section at any point of its length, and therefore is a line of curvature. Similarly, the curve of intersection of S_1 and S_3 is a line of curvature of S_1.

Ex. 1. By means of Ex. 8, § 232, prove that if two surfaces cut at a constant angle and their curve of intersection is a line of curvature of one, then it is a line of curvature of the other; also, that if the curve of intersection of two surfaces is a line of curvature on each, the two surfaces cut at a constant angle.

Ex. 2. If a line of curvature of a surface lies on a sphere, the surface and sphere cut at a constant angle at all points of the line.

Ex. 3. If the normals to a surface at all points of a plane section make a constant angle with the plane of the section, the section is a line of curvature.

235. Curvature at points of a generator of a skew surface. We have shewn that a ruled conicoid can be found to touch a given skew surface at all points of a given generator, (§ 218). If P is any point of the generator, the generators of the conicoid through P are the inflexional tangents of the skew surface, and therefore the conicoid and surface have the same indicatrix at P. Hence the sections of the conicoid and of the surface through P have the same curvature.

Ex. Investigate the principal radii of a skew surface at points of a given generator.

Take the generator as x-axis, the central point as origin, and the tangent plane at the origin as xy-plane. The equation to the conicoid which has the same principal radii is then of the form

$$2wz+2fyz+2hxy+by^2+cz^2=0.$$

Whence at $(x, 0, 0)$ we have

$$p=0, \quad q=\frac{-hx}{w}, \quad r=0, \quad s=\frac{-h}{w}, \quad t=-\frac{bw^2-2fhxw+ch^2x^2}{w^3}.$$

The principal radii are therefore given by

$$\delta^2\rho^2-\frac{\sqrt{\delta^2+x^2}}{h}(b\delta^2-2fx\delta+cx^2)\rho-(\delta^2+x^2)^2=0,$$

where δ is the parameter of distribution for the generator.

236. The measure of curvature at a point. Gauss suggested the following method of estimating the curvature of a surface at a given point. Consider a closed portion, S, of the surface whose area is A. Draw from the centre of a sphere of unit radius parallels to the normals to the surface at all points of the boundary of S. These intercept on the surface of the sphere a portion of area a, whose boundary is called the **horograph** of the portion S, and a is taken to measure the **whole curvature** of the portion S. The average curvature over S is $\frac{a}{A}$. If P is a point within S, then $\mathrm{Lt}\frac{a}{A}$ as S is indefinitely diminished is the **measure of curvature** or **specific curvature** at P.

237. Expressions for the measure of curvature. *If ρ_1 and ρ_2 are the principal radii at a point* P *the measure of curvature at* P *is* $\dfrac{1}{\rho_1\rho_2}$.

Let PQ, PR, (fig. 65), be infinitesimal arcs of the lines of curvature through P, and let QS and RS be arcs of the lines of curvature through Q and R. Then the normals to the surface at P and Q intersect at C_1, so that

$$PC_1 = QC_1 = \rho_1,$$

and the normals at P and R intersect at C_2, so that

$$PC_2 = RC_2 = \rho_2.$$

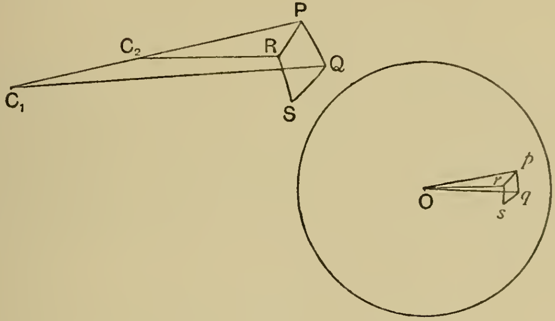

FIG. 65.

If the angles PC_1Q, PC_2R are $\delta\theta_1$ and $\delta\theta_2$, we have

$$PQ = \rho_1\delta\theta_1, \quad PR = \rho_2\delta\theta_2,$$

and the area PQRS is $\rho_1\rho_2\delta\theta_1\delta\theta_2$.

If $pqrs$ is the horograph corresponding to PQRS,

$$pq = \delta\theta_1, \quad pr = \delta\theta_2.$$

Therefore the measure of curvature at P

$$= \text{Lt}\,\frac{pqrs}{PQRS} = \text{Lt}\,\frac{\delta\theta_1\delta\theta_2}{\rho_1\rho_2\delta\theta_1\delta\theta_2} = \frac{1}{\rho_1\rho_2}.$$

Cor. The measure of curvature at a point of the surface

$$z = f(x, y) \text{ is } \frac{rt - s^2}{(1 + p^2 + q^2)^2}.$$

Ex. 1. If a cone of revolution, semivertical angle α, circumscribes an ellipsoid, the plane of contact divides the surface into two portions whose total curvatures are $2\pi(1+\sin\alpha)$, $2\pi(1-\sin\alpha)$.

The horograph is the circle of intersection of the unit sphere and the right cone whose vertex is the centre and semivertical angle $\frac{\pi}{2}-\alpha$.

Ex. 2. Any diametral plane divides an ellipsoid into two portions whose total curvatures are equal.

Ex. 3. The measure of curvature at a point P of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is $\frac{p^4}{a^2b^2c^2}$, where p is the perpendicular from the centre to the tangent plane at P.

Ex. 4. Prove that at any point P of the paraboloid $y^2+z^2=4ax$, the measure of curvature is $\frac{1}{4\mathrm{SP}^2}$, where S is the point $(a, 0, 0)$, and that the whole curvature of the portion of the surface cut off by the plane $x=x_0$ is $2\pi\left(1-\sqrt{\frac{a}{a+x_0}}\right)$.

Ex. 5. At a point of a given generator of a skew surface distant x from the central point the measure of curvature is $\frac{-\delta^2}{(\delta^2+x^2)^2}$, where δ is the parameter of distribution.

Ex. 6. If the tangent planes at any two points P and P′ of a given generator of a skew surface are at right angles, and the measures of curvature at P and P′ are R and R_1, prove that $\sqrt{R}+\sqrt{R_1}$ is constant.

Ex. 7. Find the measure of curvature at the point (x, y, z) on the surface $(y^2+z^2)(2x-1)+2x^3=0$.

Ex. 8. The binormals to a given curve generate a skew surface. Prove that its measure of curvature at a point of the curve is $-1/\sigma^2$.

Ex. 9. The normals to a skew surface at points of a generator lie on a hyperbolic paraboloid. Prove that at any point of the generator the surface and paraboloid have the same measure of curvature.

CURVILINEAR COORDINATES.

238. We have seen, (§ 185), that the equations

$$x=f_1(U, V), \quad y=f_2(U, V), \quad z=f_3(U, V),$$

where U and V are parameters, determine a surface. If we assign a particular value to one of the parameters, say U, then the locus of the point (x, y, z) as V varies is a curve on the surface, since x, y, z are now functions of one parameter. If the two curves corresponding to $U=u$, $V=v$, pass through a point P, the position of P may be

considered as determined by the values u and v of the parameters, and these values are then called the **curvilinear coordinates** of the point P. Thus a point on an ellipsoid is determined in position if the parameters of the confocal hyperboloids which pass through it are known, and these parameters may be taken as the curvilinear coordinates of the point. If one of the parameters remains constant while the other varies, the locus of the point is the curve of intersection of the ellipsoid and the hyperboloid which corresponds to the constant parameter.

Ex. 1. The helicoid is given by $x=u\cos\theta$, $y=u\sin\theta$, $z=c\theta$. What curves correspond to $u=$constant, $\theta=$constant?

Ex. 2. The hyperboloid of one sheet is given by

$$\frac{x}{a}=\frac{\lambda+\mu}{1+\lambda\mu},\quad \frac{y}{b}=\frac{1-\lambda\mu}{1+\lambda\mu},\quad \frac{z}{c}=\frac{\lambda-\mu}{1+\lambda\mu}.$$

What curves correspond to $\lambda=$constant, $\mu=$constant? If λ and μ are the curvilinear coordinates of a point on the surface, what is the locus of the point when (i) $\lambda=\mu$, (ii) $\lambda\mu=k$?

239. Direction-cosines of the normal to the surface. If O, (fig. 66), is the point of a given surface whose curvi-

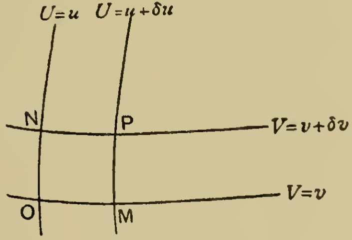

Fig. 66.

linear coordinates are u, v, the direction-cosines of the tangents at O to the curves $U=u$, $V=v$, are proportional to

$$x_v,\ y_v,\ z_v;\quad x_u,\ y_u,\ z_u.$$

Therefore if l, m, n are the direction-cosines of the normal at O to the surface,

$$lx_u+my_u+nz_u=0,$$

$$lx_v+my_v+nz_v=0.$$

and $$\frac{l}{y_u z_v - z_u y_v} = \frac{m}{z_u x_v - x_u z_v} = \frac{n}{x_u y_v - y_u x_v} = \frac{\pm 1}{\sqrt{EG - F^2}},$$

where
$$E = x_u^2 + y_u^2 + z_u^2,$$
$$F = x_u x_v + y_u y_v + z_u z_v,$$
$$G = x_v^2 + y_v^2 + z_v^2.$$

(Cf. § 185.)

Cor. If θ denotes the angle NOM,
$$\cos\theta = \frac{F}{\sqrt{EG}}.$$

240. The linear element. An equation between the curvilinear coordinates of a point on a surface represents a curve drawn on the surface. We proceed to find the relation between the differentials of the arc of the curve and the coordinates.

Let O and P, (fig. 66), be adjacent points of the curve and have cartesian coordinates
$$(x, y, z), \quad (x+\delta x, y+\delta y, z+\delta z);$$
and curvilinear coordinates
$$(u, v), \quad (u+\delta u, v+\delta v).$$

Then
$$x+\delta x = x + x_u \delta u + x_v \delta v + \dots,$$
$$y+\delta y = y + y_u \delta u + y_v \delta v + \dots,$$
$$z+\delta z = z + z_u \delta u + z_v \delta v + \dots,$$

and hence $$\mathrm{OP}^2 = E\,\delta u^2 + 2F\,\delta u\,\delta v + G\,\delta v^2,$$

if cubes and higher powers are rejected.

Therefore if ds is the differential of the arc of the curve, since $\mathrm{Lt}(\mathrm{OP}/\delta s) = 1$,
$$ds^2 = E du^2 + 2F du dv + G dv^2.$$

The value of ds given by this equation is called the **linear element** of the surface.

Ex. For the surface of revolution,
$$x = u\cos\theta, \quad y = u\sin\theta, \quad z = f(u), \quad ds^2 = (1+f'^2)du^2 + u^2 d\theta^2.$$
Find f if $ds^2 = z^2 du^2 + u^2 d\theta^2$.

241. The principal radii and lines of curvature. We can find the principal radii at a point of the surface when

the coordinates are expressed as functions of two parameters as follows:

The normal to the surface is the principal normal of any normal section, and therefore if it has direction-cosines l, m, n, we have for the normal section whose radius of curvature is ρ,

$$\frac{l}{\rho}=\frac{d^2x}{ds^2},\quad \frac{m}{\rho}=\frac{d^2y}{ds^2},\quad \frac{n}{\rho}=\frac{d^2z}{ds^2}.$$

Whence
$$\frac{1}{\rho}=\frac{ld^2x+md^2y+nd^2z}{ds^2}. \qquad\qquad (1)$$

But
$$dx=x_u du+x_v dv,$$

and
$$d^2x=x_{uu}du^2+2x_{uv}du\,dv+x_{vv}dv^2+x_u d^2u+x_v d^2v,$$

and we have similar expressions for dy, dz, d^2y and d^2z.

Again,

$$l=\frac{y_u z_v-z_u y_v}{H},\quad m=\frac{z_u x_v-x_u z_v}{H},\quad n=\frac{x_u y_v-y_u x_v}{H},$$

where
$$H^2=EG-F^2.$$

Substituting these in (1), we obtain

$$\frac{H}{\rho}=\frac{E'du^2+2F'du\,dv+G'dv^2}{Edu^2+2Fdu\,dv+Gdv^2}, \qquad\qquad (2)$$

where
$$E'\equiv\begin{vmatrix} x_{uu}, & y_{uu}, & z_{uu} \\ x_u, & y_u, & z_u \\ x_v, & y_v, & z_v \end{vmatrix},\quad F'\equiv\begin{vmatrix} x_{uv}, & y_{uv}, & z_{uv} \\ x_u, & y_u, & z_u \\ x_v, & y_v, & z_v \end{vmatrix},$$

$$G'\equiv\begin{vmatrix} x_{vv}, & y_{vv}, & z_{vv} \\ x_u, & y_u, & z_u \\ x_v, & y_v, & z_v \end{vmatrix}.$$

Equation (2) may be written

$$du^2(EH-E'\rho)+2du\,dv(FH-F'\rho)+dv^2(GH-G'\rho)=0, \quad (3)$$

and gives two values of $du:dv$ which correspond to the two normal sections with a given radius of curvature. If ρ is a principal radius the values coincide. Therefore the principal radii are given by

$$\rho^2(E'G'-F'^2)-H\rho(EG'+GE'-2FF')+H^4=0. \quad (4)$$

If ρ is a principal radius, by (3) and (4),

$$\frac{Edu+Fdv}{E'du+F'dv}=\frac{Fdu+Gdv}{F'du+G'dv}=\frac{\rho}{H},$$

and therefore for a line of curvature,

$$du^2(EF'-E'F)-du\,dv(GE'-G'E)+dv^2(FG'-F'G)=0. \quad (5)$$

Cor. 1. The measure of curvature is $\dfrac{E'G'-F'^2}{H^4}$.

Cor. 2. For an umbilic $\dfrac{E}{E'}=\dfrac{F}{F'}=\dfrac{G}{G'}$.*

Equation (2) may also be obtained as follows:

If O, (x, y, z) is the point considered, and P, $(x+\delta x, y+\delta y, z+\delta z)$ is an adjacent point on a normal section through O, ρ, the radius of curvature of the section, is given by

$$\rho=\text{Lt}\,\frac{OP^2}{2p},$$

where p is the perpendicular from P to the tangent plane at O. The equation to the tangent plane is $\Sigma(\xi-x)(y_u z_v-z_u y_v)=0$, and

$$x+\delta x=x+(x_u\delta u+x_v\delta v)+\tfrac{1}{2}(x_{uu}\delta u^2+2x_{uv}\delta u\,\delta v+x_{vv}\delta v^2)\text{; etc.}$$

Hence
$$\begin{aligned} p&=\frac{\Sigma\,\delta x(y_u z_v-z_u y_v)}{\sqrt{\Sigma(y_u z_v-z_u y_v)^2}}\\ &=\frac{\Sigma(x_{uu}\delta u^2+2x_{uv}\delta u\,\delta v+x_{vv}\delta v^2)(y_u z_v-z_u y_v)}{2H}\\ &=\frac{E'\delta u^2+2F'\delta u\,\delta v+G'\delta v^2}{2H}.\end{aligned}$$

Therefore, since $\text{Lt}\,OP^2=Edu^2+2Fdu\,dv+Gdv^2$,

$$\frac{H}{\rho}=\frac{E'du^2+2F'du\,dv+G'dv^2}{Edu^2+2Fdu\,dv+Gdv^2}.$$

Ex. 1. Find the principal radii and lines of curvature of the surface $z=f(x, y)$.

Take $u\equiv x$, $v\equiv y$, then

$$x_u=1,\ y_u=0,\ z_u=p;\quad x_v=0,\ y_v=1,\ z_v=q;$$
$$x_{uu}=y_{uu}=x_{uv}=y_{uv}=0,$$
$$z_{uu}=r,\ z_{uv}=s,\ z_{vv}=t.$$

Hence $E=1+p^2$, $F=pq$, $G=1+q^2$, $H^2=EG-F^2=1+p^2+q^2$;
$E'=r$, $F'=s$, $G'=t$;

and on substituting in equations (4) and (5), we obtain the equations of § 232.

* The student will find the methods of curvilinear coordinates discussed and applied in a recent treatise on *Differential Geometry* by L. P. Eisenhart, (Ginn & Co.). He is also referred to *Applications Géométriques du Calcul Différentiel*, W. de Tannenberg; *Théorie des Surfaces*, Darboux; *Geometria Differenziale*, Bianchi.

Ex. 2. A ruled surface is generated by the binormals of a given curve. Find the principal radii at a point distant r from the curve.

The coordinates of the point are given by

$$\xi = x + l_3 r, \quad \eta = y + m_3 r, \quad \zeta = z + n_3 r,$$

and are functions of s and r. Taking $u \equiv s$ and $v \equiv r$, and applying Frenet's formulae, we obtain

$$E = 1 + \frac{r^2}{\sigma^2}, \quad F = 0, \quad G = 1, \quad H = \sqrt{1 + \frac{r^2}{\sigma^2}};$$

$$E' = \frac{r\sigma'}{\sigma^2} - \frac{1}{\rho} - \frac{r^2}{\rho\sigma^2}, \quad F' = -\frac{1}{\sigma}, \quad G' = 0.$$

Therefore the principal radii are given by

$$\frac{R^2}{\sigma^2} - R\sqrt{1 + \frac{r^2}{\sigma^2}}\left\{\frac{1}{\rho}\left(1 + \frac{r^2}{\sigma^2}\right) - \frac{r\sigma'}{\sigma^2}\right\} - \left(1 + \frac{r^2}{\sigma^2}\right)^2 = 0.$$

Ex. 3. Find the measure of curvature at the line of striction.

The curve is the line of striction, and when $r = 0$, the measure of curvature is $-1/\sigma^2$.

Ex. 4. Apply the method of curvilinear coordinates to prove that the principal radius of a developable at a distance l along a generator from the edge of regression is $\dfrac{l\sigma}{\rho}$.

Ex. 5. Apply the method of curvilinear coordinates to prove that for the helicoid $x = u\cos\theta$, $y = u\sin\theta$, $z = c\theta$, $\rho_1 = -\rho_2 = \dfrac{u^2 + c^2}{c}$, and that the lines of curvature are given by $d\theta = \dfrac{\pm du}{\sqrt{u^2 + c^2}}$.

Ex. 6. Find the locus of points on the helicoid at which the measure of curvature has a given value.

Ex. 7. For the surface

$$\frac{x}{a} = \frac{u + v}{2}, \quad \frac{y}{b} = \frac{u - v}{2}, \quad z = \frac{uv}{2},$$

prove that the principal radii are given by

$$a^2b^2\rho^2 + kab\rho(a^2 - b^2 + uv) - k^4 = 0,$$

where
$$4k^2 \equiv 4a^2b^2 + a^2(u - v)^2 + b^2(u + v)^2,$$

and that the lines of curvature are given by

$$\frac{du}{\sqrt{a^2 + b^2 + u^2}} = \frac{\pm dv}{\sqrt{a^2 + b^2 + v^2}}.$$

Ex. 8. For the surface

$$x = 3u(1 + v^2) - u^3, \quad y = 3v(1 + u^2) - v^3, \quad z = 3u^2 - 3v^2,$$

the principal radii at any point are

$$\pm \tfrac{3}{2}(1 + u^2 + v^2)^2,$$

and the lines of curvature are given by $u = a_1$, $v = a_2$, where a_1 and a_2 are arbitrary constants.

Ex. 9. The squares of the semi-axes of the confocals through a point P on the conicoid

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=1$$

are $a_1 \equiv a-\lambda$, $b_1 \equiv b-\lambda$, $c_1 \equiv c-\lambda$; $a_2 \equiv a-\mu$, $b_2 \equiv b-\mu$, $c_2 \equiv c-\mu$.

Taking λ and μ as the curvilinear coordinates of the point, prove that

$$4E=\frac{\lambda(\lambda-\mu)}{a_1b_1c_1}, \qquad F=0, \quad 4G=\frac{\mu(\mu-\lambda)}{a_2b_2c_2};$$

$$\frac{4E'}{H}=-\sqrt{\frac{abc}{\lambda\mu}}\,\frac{(\lambda-\mu)}{a_1b_1c_1}, \quad F'=0, \quad \frac{4G'}{H}=\sqrt{\frac{abc}{\lambda\mu}}\,\frac{(\lambda-\mu)}{a_2b_2c_2}.$$

Deduce that

$$\rho_1=\sqrt{\frac{\lambda^3\mu}{abc}} \quad\text{and}\quad \rho_2=\sqrt{\frac{\lambda\mu^3}{abc}},$$

and that the lines of curvature are $\lambda=$constant, $\mu=$constant.

If l, m, n are the direction-cosines of the normal to the surface,

$$\frac{E'}{H}=\Sigma l x_{uu}=-\Sigma l_u x_u,$$

since

$$\Sigma l x_u=0.$$

We have also

$$l=\frac{px}{a}=\sqrt{\frac{abc}{\lambda\mu}}\,\frac{x}{a}, \text{ etc.}$$

Ex. 10. Prove that if F and F' are zero, the parametric curves are lines of curvature.

Examples XIII.

1. Prove that along a given line of curvature of a conicoid, one principal radius varies as the cube of the other.

2. Prove that the principal radii at a point of the surface $xyz=a^3$ are given by

$$\rho^2+\frac{2\rho}{p}(x^2+y^2+z^2)+\frac{27a^6}{p^4}=0,$$

where p is the perpendicular from the origin to the tangent plane at the point. Shew that this equation can be written in the form

$$\frac{x^2}{3x^2+p\rho}+\frac{y^2}{3y^2+p\rho}+\frac{z^2}{3z^2+p\rho}=1,$$

and that if (ξ, η, ζ) is a centre of principal curvature at (x, y, z),

$$\frac{x}{2x-\xi}+\frac{y}{2y-\eta}+\frac{z}{2z-\zeta}=3.$$

3. Find the principal radii of the surface $a^2x^2=z^2(x^2+y^2)$ at the points where $x=y=z$.

4. Prove that the cone

$$kxy=z(\sqrt{x^2+z^2}+\sqrt{y^2+z^2})$$

passes through a line of curvature of the paraboloid $xy=az$.

5. For the surface

$$x=u\cos\theta,\quad y=u\sin\theta,\quad z=f(\theta),$$

prove that the angles that the lines of curvature make with the generators are given by

$$\tan^2\phi+\frac{f''}{f'}\frac{u}{\sqrt{u^2+f'^2}}\tan\phi-1=0.$$

6. For a rectangular hyperboloid, (in which the asymptotic cone has three mutually perpendicular generators), the normal chord at any point is the harmonic mean between the principal radii.

7. PT is tangent at P to a curve on an ellipsoid along which the measure of curvature is constant. Prove that the normal section of the ellipsoid through PT is an ellipse which has one of its vertices at P.

8. Prove that at a point of the intersection of the cylindroid $z(x^2+y^2)=mxy$ and the cylinder $(x^2+y^2)^3=a^2(x^2-y^2)^2$ the measure of curvature of the former varies as $\dfrac{1}{x^2+y^2}$.

9. The principal radii at a point P of a surface are ρ_1 and ρ_2 and the radius of curvature of a normal section through P is R. Shew that the normal to the surface at a neighbouring point Q on the section distant s from P, makes with the principal normal to the section at Q an angle

$$s\{(\rho_1^{-1}-R^{-1})(R^{-1}-\rho_2^{-1})\}^{\frac{1}{2}}.$$

10. Prove that the lines of curvature of the paraboloid $xy=az$ lie on the surfaces

$$\sinh^{-1}\frac{x}{a}\pm\sinh^{-1}\frac{y}{a}=A,$$

where A is an arbitrary constant.

11. Shew that the sum or difference of the distances of any point on a line of curvature of the paraboloid $xy=az$ from the generators through the vertex is constant.

12. A curve is drawn on the surface

$$2z=rx^2+2sxy+ty^2$$

touching the axis of x at the origin and with its osculating plane inclined to the z-axis at an angle ϕ. Prove that at the origin

$$x''=0,\quad y''=r\tan\phi,\quad z''=r,\quad x'''=-r^2\sec^2\phi,\quad z'''=3rs\tan\phi.$$

13. Prove that the whole curvature of the portion of the paraboloid $xy=az$ bounded by the generators through the origin and through the point (x, y, z) is

$$-\tan^{-1}\frac{z^2}{\sqrt{y^2z^2+z^2x^2+x^2y^2}}.$$

14. Prove that the differential equation of the projections on the xy-plane of the lines of curvature of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

is $$\beta b^2xy\,dx^2-(\beta b^2x^2+\alpha a^2y^2+a^2b^2\gamma)dx\,dy+\alpha a^2xy\,dy^2=0,$$

where $$\alpha\equiv b^2-c^2,\quad \beta\equiv c^2-a^2,\quad \gamma\equiv a^2-b^2.$$

Deduce that

$$(\alpha a^2p^2-\beta b^2)\left(xy\frac{dp}{dx}+p^2x-py\right)=0,$$

where $p\equiv\frac{dy}{dx}$, and hence shew that the integral of the equation is

$$x^2-\frac{y^2}{k}=\frac{\gamma a^2b^2}{\alpha a^2k-\beta b^2},$$

where k is an arbitrary constant.

Prove that if $k=\frac{\beta b^2(b^2-\lambda)}{\alpha a^2(a^2-\lambda)}$ this reduces to

$$\frac{x^2(c^2-a^2)}{a^2(a^2-\lambda)}-\frac{y^2(b^2-c^2)}{b^2(b^2-\lambda)}=1,$$

and deduce that the lines of curvature are the curves of intersection of the ellipsoid and its confocals.

15. Prove that the measure of curvature at points of a generator of a skew surface varies as $\cos^4\theta$, where θ is the angle between the tangent planes at the point and at the central point.

16. Prove that the surface

$$4a^2z^2=(x^2-2a^2)(y^2-2a^2)$$

has a line of umbilics lying on the sphere $x^2+y^2+z^2=4a^2$.

17. A ruled surface is generated by the principal normals to a given curve; prove that at the point of a principal normal distant r from the curve the principal radii are given by

$$\frac{R^2}{\sigma^2}+R\sqrt{\left(1-\frac{r}{\rho}\right)^2+\frac{r^2}{\sigma^2}}\left[\frac{r\sigma'}{\sigma^2}-\frac{r^2}{\rho\sigma}\left(\frac{\sigma'}{\sigma}+\frac{\rho'}{\rho}\right)\right]-\left[\left(1-\frac{r}{\rho}\right)^2+\frac{r^2}{\sigma^2}\right]^2=0.$$

What are the principal radii at points of the curve?

18. If l, m, n are the direction-cosines of the normal at a point to the surface $z=f(x, y)$ the equation for the principal radii can be written

$$\frac{1}{\rho^2}-\frac{1}{\rho}\left(\frac{\partial l}{\partial x}+\frac{\partial m}{\partial y}\right)+\frac{\partial(l, m)}{\partial(x, y)}=0.$$

19. Prove that the osculating plane of the line of curvature of the surface

$$2z=\frac{x^2}{\rho_1}+\frac{y^2}{\rho_2}+\frac{1}{3}(ax^3+3bx^2y+3cxy^2+dy^3)\ldots,$$

which touches **OX**, makes an angle ϕ with the plane **ZOX**, such that

$$\tan\phi=\frac{\rho_1{}^2\rho_2 b}{\rho_2-\rho_1}.$$

20. Prove that for the surface formed by the revolution of the tractrix about its axis

$$z=a\left(\log\tan\frac{\phi}{2}+\cos\phi\right),\quad u=a\sin\phi,$$

and that the surface has at any point a constant measure of curvature $-a^{-2}$.

21. If the surface of revolution

$$x = u\cos\theta,\quad y = u\sin\theta,\quad z = f(u)$$

is a minimal surface, $f'(1+f'^2)+uf''=0$.

Hence, shew that the only real minimal surface of revolution is formed by the revolution of a catenary about its directrix.

22. At a point of the curve of intersection of the paraboloid $xy = cz$ and the hyperboloid $x^2+y^2-z^2+c^2=0$ the principal radii of the paraboloid are $\dfrac{z^2}{c}(1\pm\sqrt{2})$.

23. The principal radius of a cone at any point of its curve of intersection with a concentric sphere varies as $(\sin\lambda\sin\mu)^{\frac{3}{2}}$, where λ and μ are the angles that the generator through the point makes with the focal lines.

24. A straight line drawn through the variable point

$$\mathsf{P}, (a\cos\phi,\ a\sin\phi,\ 0),$$

parallel to the zx-plane makes an angle θ, where θ is some function of ϕ, with the z-axis. Prove that the measure of curvature at P of the surface generated by the line is

$$-\frac{\cos^2\phi}{a^2(1-\sin^2\theta\sin^2\phi)^2}\left(\frac{d\theta}{d\phi}\right)^2.$$

25. A variable ellipsoid whose axes are the coordinate axes touches the given plane $px+qy+rz=1$. Prove that the locus of the centres of principal curvature at the point of contact is

$$(px+qy+rz-1)(p^3yz+q^3zx+r^3xy)=xyz(p^2+q^2+r^2)^2.$$

CHAPTER XVII.

ASYMPTOTIC LINES.

242. A curve drawn on a surface so as to touch at each point one of the inflexional tangents through the point is called an **asymptotic line** on the surface.

243. The differential equation of asymptotic lines. If l_1, m_1, n_1 are the direction-cosines of the tangent to an asymptotic line on the surface $z=f(x, y)$, we have, from § 181,

$$rl_1{}^2+2sl_1m_1+tm_1{}^2=0;$$

whence, as in § 232, the differential equation of the projections on the xy-plane of the asymptotic lines is

$$rdx^2+2sdxdy+tdy^2=0.$$

It is evident from the definition that the asymptotic lines of a hyperboloid of one sheet are the generators. This may be easily verified from the differential equation. If the equation to the hyperboloid is $x^2/a^2+y^2/b^2-z^2/c^2=1$,

$$p=\frac{c^2x}{a^2z},\quad q=\frac{c^2y}{b^2z},\quad r=\frac{-c^4}{a^2z^3}\left(1-\frac{y^2}{b^2}\right),\quad s=\frac{-c^4xy}{a^2b^2z^3},$$
$$t=\frac{-c^4}{b^2z^3}\left(1-\frac{x^2}{a^2}\right).$$

Whence the differential equation becomes

$$\frac{dx^2}{a^2}\left(1-\frac{y^2}{b^2}\right)+2dxdy\frac{xy}{a^2b^2}+\frac{dy^2}{b^2}\left(1-\frac{x^2}{a^2}\right)=0,$$

or
$$y=y_1x\pm\sqrt{a^2y_1{}^2+b^2},\quad \text{where } y_1\equiv\frac{dy}{dx}.$$

This equation is clearly satisfied by the tangents to the ellipse $z=0$, $x^2/a^2+y^2/b^2=1$, or the projections of the

asymptotic lines are the tangents. We also have proved, (§ 104), that the projections of the generators are the tangents.

244. The osculating plane of an asymptotic line. If l_1, m_1, n_1 are the direction-cosines of the tangent to an asymptotic line,

$$pl_1+qm_1-n_1=0.$$

Therefore, by Frenet's formulae,

$$\frac{pl_2+qm_2-n_2}{\rho}=-(rl_1{}^2+2sl_1m_1+tm_1{}^2)=0.$$

Whence $$\frac{p}{m_1n_2-m_2n_1}=\frac{q}{n_1l_2-n_2l_1}=\frac{-1}{l_1m_2-l_2m_1},$$

or $$\frac{p}{l_3}=\frac{q}{m_3}=\frac{-1}{n_3}.$$

Therefore the binormal of the asymptotic line is the normal to the surface, or the tangent plane to the surface is the osculating plane of the asymptotic line.

Cor. 1. The two asymptotic lines through any point have the same osculating plane.

Cor. 2. The normals to a surface at points of an asymptotic line generate a skew surface whose line of striction is the asymptotic line.

245. The torsion of an asymptotic line. Consider the asymptotic lines through the origin on the surface

$$2z=\frac{x^2}{\rho_1}+\frac{y^2}{\rho_2}+\dots.$$

The tangents make angles $\pm\alpha$ with the x-axis, where $\tan\alpha=\sqrt{\dfrac{-\rho_2}{\rho_1}}$. Hence, for one asymptotic line,

$$l_1=\cos\alpha,\qquad m_1=\sin\alpha,\quad n_1=0$$

and $$l_2=-\sin\alpha,\quad m_2=\cos\alpha,\quad n_2=0.$$

Also, from § 244, $$l_3=\frac{-p}{\sqrt{1+p^2+q^2}}.$$

Therefore if $d\beta$ is the differential of the arc of the line,

$$\frac{l_2}{\sigma}=\frac{-(rl_1+sm_1)}{\sqrt{1+p^2+q^2}}-p\frac{d}{d\beta}(1+p^2+q^2)^{-\frac{1}{2}}.$$

But at the origin, $r=1/\rho_1$, $s=0$, $p=0$, $q=0$.

Therefore $$\frac{\sin\alpha}{\sigma}=\frac{\cos\alpha}{\rho_1},$$

or $$\frac{1}{\sigma}=\frac{1}{\rho_1}\cot\alpha=\sqrt{\frac{-1}{\rho_1\rho_2}}.$$

Cor. The two asymptotic lines through a point have the same torsion.

The asymptotic lines of a developable surface are the generators, and the osculating plane of an asymptotic line is the same at all points of the line. Hence $1/\sigma=0$. But one of the principal curvatures is also zero, and thus the equation $\frac{1}{\sigma}=\sqrt{\frac{-1}{\rho_1\rho_2}}$ is verified for developable surfaces.

For a hyperboloid of one sheet, the asymptotic lines are also the generators, but the osculating plane of an asymptotic line is not the same at all points of the line. The osculating plane at each point of the line is determinate, however, and $\frac{1}{\sigma}$ has a definite value $\sqrt{\frac{-1}{\rho_1\rho_2}}$ at each point. The value is the rate at which the tangent plane is twisting round the generator. We have thus an instance of a straight line with a definite osculating plane at each point.

Ex. 1. Prove that the projections of the asymptotic lines of the paraboloid $2z=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ on the xy-plane are given by $\frac{x}{a}\pm\frac{y}{b}=\lambda$, where λ is an arbitrary constant.

Ex. 2. Find the differential equation to the projections of the asymptotic lines of the conoid

$$x=u\cos\theta,\quad y=u\sin\theta,\quad z=f(\theta).$$

Using the values of p, q, r, s, t, given in Ex. 12, § 232, we obtain

$$\text{(i) } d\theta=0, \quad\text{or}\quad \text{(ii) } \frac{2du}{u}=\frac{z''}{z'}d\theta.$$

From (i) $\theta=\alpha$, where α is arbitrary, and hence one asymptotic line through each point is the generator.
From (ii) $u^2=\lambda z'$, where λ is arbitrary.

Ex. 3. Prove that the asymptotic lines of the helicoid

$$x=u\cos\theta,\quad y=u\sin\theta,\quad z=c\theta$$

consist of the generators and the curves of intersection with coaxal right cylinders.

Ex. 4. Prove that the projection on the xy-plane of an asymptotic line of the cylindroid

$$x=u\cos\theta,\quad y=u\sin\theta,\quad z=m\sin 2\theta$$

is a lemniscate.

Ex. 5. Prove that the projections on the xy-plane of the asymptotic lines of the conoid

$$x=u\cos\theta,\quad y=u\sin\theta,\quad z=ae^{m\theta}$$

are equiangular spirals.

Ex. 6. Prove that the differential equation to the projections on the xy-plane of the asymptotic lines of the surface of revolution

$$x=u\cos\theta,\quad y=u\sin\theta,\quad z=f(u)$$

is
$$z''du^2+uz'd\theta^2=0,\ \text{where}\ z'\equiv\frac{dz}{du},\quad z''\equiv\frac{d^2z}{du^2}.$$

Ex. 7. Find the asymptotic lines of the cone $z=u\cot\alpha$.

Ex. 8. For the hyperboloid of revolution $\frac{u^2}{a^2}-\frac{z^2}{c^2}=1$, prove that the projections of the asymptotic lines on the xy-plane are given by

$$u=a\sec(\theta-\alpha),$$

where α is an arbitrary constant.

Ex. 9. The asymptotic lines of the catenoid $u=c\cosh\frac{z}{c}$ lie on the cylinders $2u=c(ae^{\theta}+a^{-1}e^{-\theta})$, where a is arbitrary.

Ex. 10. Find the curvatures of the asymptotic lines through the origin on the surface

$$2z=\frac{x^2}{\rho_1}+\frac{y^2}{\rho_2}+\frac{1}{3}(ax^3+3bx^2y+3cxy^2+dy^3)+\dots.$$

Differentiating $rl_1^2+2sl_1m_1+tm_1^2=0$, we obtain

$$\frac{-2}{\rho}\{rl_1l_2+s(l_1m_2+l_2m_1)+tm_1m_2\}$$
$$=l_1^3\frac{\partial r}{\partial x}+l_1m_1\left(l_1\frac{\partial r}{\partial y}+2l_1\frac{\partial s}{\partial x}+2m_1\frac{\partial s}{\partial y}+m_1\frac{\partial t}{\partial x}\right)+m_1^3\frac{\partial t}{\partial y}.\quad\dots(1)$$

For one line, $l_1=\cos\alpha$, $m_1=\sin\alpha$, $n_1=0$; $l_2=-\sin\alpha$, $m_2=\cos\alpha$, $n_2=0$, where $\tan^2\alpha=\frac{-\rho_2}{\rho_1}$. And at the origin, $r=\frac{1}{\rho_1}$, $s=0$, $t=\frac{1}{\rho_2}$, $\frac{\partial r}{\partial x}=a$, $\frac{\partial r}{\partial y}=\frac{\partial s}{\partial x}=b$, $\frac{\partial s}{\partial y}=\frac{\partial t}{\partial x}=c$, $\frac{\partial t}{\partial y}=d$. Whence (1) becomes

$$\frac{1}{\rho}=\frac{-\rho_1\rho_2}{2(\rho_1-\rho_2)^{\frac{3}{2}}}\left\{\frac{a\rho_1}{\sqrt{-\rho_2}}+3b\sqrt{\rho_1}+3c\sqrt{-\rho_2}-\frac{d\rho_2}{\sqrt{\rho_1}}\right\}.$$

To obtain the curvature of the other line we must change the sign of $\sqrt{-\rho_2}$.

Ex. 11. The normals to a surface at points of an asymptotic line generate a skew surface, and the two surfaces have the same measure of curvature at any point of the line.

Ex. 12. In curvilinear coordinates the differential equation to the asymptotic lines is

$$E'du^2+2F'du\,dv+G'dv^2=0.$$

Apply the method of curvilinear coordinates to Exs. 3, 4, 6.

Ex. 13. Prove that the asymptotic lines of the surface

$$x=v-2u-e^{-u},\quad y=e^{v-u},\quad z=e^u-v$$

lie on the cylinders

$$zy+ay-e^a=0,\quad xy+by+e^{-b}=0,$$

where a and b are arbitrary constants.

Ex. 14. For the surface

$$\frac{x}{a}=\frac{u+v}{2},\quad \frac{y}{b}=\frac{u-v}{2},\quad z=\frac{uv}{2},$$

the asymptotic lines are given by $u=\lambda$, $v=\mu$, where λ and μ are arbitrary constants.

Ex. 15. For the surface

$$x=3u(1+v^2)-u^3,\quad y=3v(1+u^2)-v^3,\quad z=3u^2-3v^2,$$

the asymptotic lines are $u\pm v=$constant.

Ex. 16. Prove that the asymptotic lines on the surface of revolution

$$x=u\cos\theta,\quad y=u\sin\theta,\quad z=f(u),$$

where
$$z=a\left(\log\tan\frac{\phi}{2}+\cos\phi\right)\text{ and }u=a\sin\phi$$

are given by
$$d\theta=\pm\frac{d\phi}{\sin\phi}.$$

GEODESICS.

246. A curve drawn on a surface so that its osculating plane at any point contains the normal to the surface at the point is a **geodesic**. It follows that the principal normal at any point is the normal to the surface.

An infinitesimal arc PQ of a geodesic coincides with the section of the surface by the osculating plane at P; that is, with a normal section through P. Therefore, by Meunier's theorem, the geodesic arc PQ is the arc of least curvature through P and Q, or the shortest distance on the surface between two adjacent points P and Q is along the geodesic through the points.

Ex. 1. The principal normal to a right helix is the normal to the cylinder, and hence the geodesics on a cylinder are the helices that can be drawn on it.

Ex. 2. If a geodesic is either a plane curve or a line of curvature, it is both. (Apply § 230.)

247. Geodesics on developable surfaces. If the surface is a developable, the infinitesimal arc PQ is unaltered in length when the surface is developed into a plane. Therefore if a geodesic passes through two points A and B of a developable, and the surface is developed into a plane, the geodesic develops into the straight line joining the points A and B in the plane.

Ex. 1. The geodesics on any cylinder are helices.

When the cylinder is developed into a plane, any helix develops into a straight line.

Ex. 2. An infinite number of geodesics can be drawn through two points A and B of a cylinder.

If any number of sheets is unwrapped from the cylinder and A′, A″, A‴, ... , B′, B″, B‴, ... are the positions of A, B on the plane so formed, the line joining any one of the points A′, A″, A‴, ... to any one of the points B′, B″, B‴, ... becomes a geodesic when the sheets are wound again on the cylinder.

Ex. 3. If the cylinder is $x^2+y^2=a^2$, and A and B are

$$(a, 0, 0), \quad (a\cos\alpha, a\sin\alpha, b),$$

the geodesics through A and B are given by

$$x=a\cos\theta, \quad y=a\sin\theta, \quad z=\frac{b\theta}{2n\pi+\alpha}.$$

248. The differential equations to geodesics. From the definition of a geodesic, we have

$$\frac{\dfrac{d^2x}{ds^2}}{F_x}=\frac{\dfrac{d^2y}{ds^2}}{F_y}=\frac{\dfrac{d^2z}{ds^2}}{F_z} \quad \text{.........................(1)}$$

for geodesics on the surface $F(x, y, z)=0$, and

$$\frac{\dfrac{d^2x}{ds^2}}{p}=\frac{\dfrac{d^2y}{ds^2}}{q}=\frac{\dfrac{d^2z}{ds^2}}{-1} \quad \text{.........................(2)}$$

for geodesics on the surface $z=f(x, y)$.

If an integral of one of the equations (1) can be found, it will contain two arbitrary constants, and with the equation to the surface, $F(x, y, z)=0$, will represent the geodesics.

Similarly, an integral of one of the equations (2) and the equation $z=f(x, y)$ together represent the geodesics of the surface $z=f(x, y)$.

Ex. 1. Find the equations to the geodesics on the helicoid

$$x=u\cos\theta,\quad y=u\sin\theta,\quad z=c\theta.$$

For a geodesic, $qx''-py''=0$, and $p=-\dfrac{\sin\theta}{u}\dfrac{dz}{d\theta},\quad q=\dfrac{\cos\theta}{u}\dfrac{dz}{d\theta}$;

therefore $$x''\cos\theta+y''\sin\theta=0$$

or $$u''-u\theta'^2=0. \quad \text{.........................(1)}$$

But $$x'^2+y'^2+z'^2=1\,;$$

therefore $$u'^2+(u^2+c^2)\theta'^2=1. \quad \text{.........................(2)}$$

Hence, from (1), $$u(1-u'^2)=(u^2+c^2)u'',$$

which gives, on integrating,

$$1-u'^2=\frac{k^2}{u^2+c^2},$$

where k is an arbitrary constant.

Eliminating ds between this equation and equation (2), we obtain a first integral

$$d\theta=\frac{\pm k\,du}{\sqrt{(u^2+c^2)(u^2+c^2-k^2)}},$$

whence the complete integral can be found in terms of elliptic functions.

Ex. 2. Find the differential equation to the projections on the xy-plane of the geodesics on the surface $z=f(x, y)$.

If l_1, m_1, n_1 are the direction-cosines of the tangent to a geodesic, and $\dfrac{1}{\rho}$ is its curvature,

$$\frac{1}{\rho}=\frac{rl_1{}^2+2sl_1m_1+tm_1{}^2}{\sqrt{1+p^2+q^2}}. \quad (\S\ 225)$$

But by Ex. 10, § 204, the radius of curvature of the projection on the xy-plane is $\rho\dfrac{(1-n_1{}^2)^{\frac{3}{2}}}{n_3}$.

And $$n_3=l_1m_2-l_2m_1=\frac{pm_1-ql_1}{\sqrt{1+p^2+q^2}}.$$

Therefore the radius of curvature of the projection is

$$\frac{(1+p^2+q^2)(l_1{}^2+m_1{}^2)^{\frac{3}{2}}}{(pm_1-ql_1)(rl_1{}^2+2sl_1m_1+tm_1{}^2)}.$$

Hence, at any point of the projection we have

$$\frac{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}=\frac{(1+p^2+q^2)\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\left(p\dfrac{dy}{dx}-q\right)\left\{r+2s\dfrac{dy}{dx}+t\left(\dfrac{dy}{dx}\right)^2\right\}}$$

or $$\frac{d^2y}{dx^2}(1+p^2+q^2)=\left\{r+2s\frac{dy}{dx}+t\left(\frac{dy}{dx}\right)^2\right\}\left(p\frac{dy}{dx}-q\right).$$

249. Geodesics on a surface of revolution. The equation to any surface of revolution is of the form

$$z = f(\sqrt{x^2 + y^2}) \quad \text{or} \quad z = f(u).$$

Hence $$p = \frac{x}{u} f' \quad \text{and} \quad q = \frac{y}{u} f'.$$

But for a geodesic, $$p \frac{d^2 y}{ds^2} = q \frac{d^2 x}{ds^2}.$$

Therefore

$$y \frac{d^2 x}{ds^2} - x \frac{d^2 y}{ds^2} = 0 \quad \text{or} \quad \frac{d}{ds}\left(y \frac{dx}{ds} - x \frac{dy}{ds}\right) = 0.$$

Hence $$y \frac{dx}{ds} - x \frac{dy}{ds} = -c,$$

where c is an arbitrary constant.

Change to polar coordinates, where

$$x = u \cos\theta, \qquad y = u \sin\theta,$$

$$x' = u' \cos\theta - u\theta' \sin\theta, \quad y' = u' \sin\theta + u\theta' \cos\theta,$$

and we get $$u^2 \frac{d\theta}{ds} = c.$$

Ex. 1. If a geodesic on a surface of revolution cuts the meridian at any point at an angle ϕ, $u \sin\phi$ is constant, where u is the distance of the point from the axis.

We have $\sin\phi = u \dfrac{d\theta}{ds}$, whence the result is simply another form of that of § 249.

Ex. 2. Deduce that on a right cylinder the geodesics are helices.

Ex. 3. The perpendiculars from the vertex of a right cone to the tangents to a given geodesic are of constant length.

If O is the vertex, the perpendicular on to the tangent at a point P

$$= \mathrm{OP} \sin\phi = u \operatorname{cosec}\alpha \sin\phi.$$

Ex. 4. Investigate the geodesics through two given points on a right cone.

Let the points be A and B, (fig. 67), and take the zx-plane through A. Let the semivertical angle of the cone be α and the plane BOz make an angle β with the zx-plane. Suppose that A and B are distant a and b from the vertex.

If the cone is slit along OA and developed into a plane, the distance of the vertex from any tangent to the geodesic remains unaltered, and therefore the geodesic develops into a straight line, (cf. § 247). Figures 67 and 68 represent the cone and its development into a plane. The circular sections of the cone through A and B become arcs of concentric circles of radii a and b, and

$$\angle \mathrm{A_1OD_1} = \frac{\text{arc } \mathrm{A_1D_1}}{\mathrm{O_1A_1}} = \frac{\text{arc } \mathrm{AD}}{\mathrm{OA}} = \beta \sin\alpha \equiv \gamma, \text{ say}.$$

The geodesic develops into A_1B_1, and if P, any point on A_1B_1, has polar coordinates r, ψ referred to O_1A_1 as initial line,

since $$\triangle O_1A_1P_1 + \triangle O_1P_1B_1 = \triangle O_1A_1B_1,$$
$$ar\sin\psi + br\sin(\gamma - \psi) = ab\sin\gamma.$$

Now the relations between the cylindrical coordinates u, θ in space and the polar coordinates r, ψ in the plane are
$$u = r\sin\alpha, \quad \psi = \theta\sin\alpha,$$
and therefore the coordinates of any point of the geodesic satisfy the equation
$$u\{a\sin(\theta\sin\alpha) + b\sin(\gamma - \theta\sin\alpha)\} = ab\sin\alpha\sin\gamma. \quad \text{.........(1)}$$

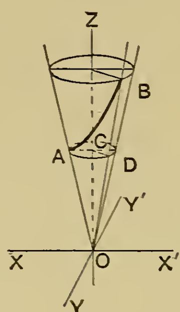

FIG. 67.

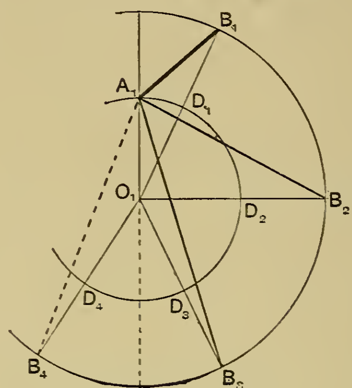

FIG. 68.

This equation represents a cylinder which intersects the cone in the geodesic.

If the arcs D_1D_2, D_2D_3, ... are each equal to the circumference of the circle in the plane ADC, the positions of OB, when in addition to the curved sector OAB of the surface of the cone, one, two, ... complete sheets are successively developed into a plane, are O_1B_2, O_1B_3 .... If A_1 and B_2 are joined and the plane sector $A_1O_1B_2$ is wrapped again on the cone, A_1B_2 becomes a second geodesic passing through A and B and completely surrounding the cone. Similarly A_1B_3 becomes a third geodesic. A_1B_4, however, does not lie on the sheets that have been unrolled from the cone, and hence the only geodesics, (in our figure), through A and B are those which develop into A_1B_1, A_1B_2, A_1B_3.

It is clear from the figure, that if $(n+1)$ geodesics pass through two points A and B, and the angle between the planes through the axis of the cone and A and B is β,
$$\sin\alpha(\beta + 2n\pi) < \pi.$$

The equations to surfaces through all the geodesics through A and B can be obtained from equation (1) by writing $(\beta + 2p\pi)\sin\alpha$ for γ.

If A and B are points on the same generator of the cone, $\beta = 2\pi$. So that, if we are to have any geodesic through A and B,
$$\sin\alpha \,.\, \beta < \pi \text{ or } \sin\alpha < \tfrac{1}{2}.$$

A geodesic on a cone will therefore not cross a generator at two points unless the semivertical angle of the cone is less than $\frac{\pi}{6}$.

Ex. 5. Find the length of the geodesic AB.

Ans. $AB^2 = a^2 + b^2 - 2ab\cos(\beta \sin \alpha)$.

Ex. 6. Find the distance of the vertex from any tangent to the geodesic AB.

Ans. $\dfrac{ab\sin(\beta\sin\alpha)}{AB}$.

Ex. 7. If A and B are points on the same generator OAB of a cone semivertical angle α, and a geodesic through A and B cuts OA at right angles at A, then $\sin\alpha < \frac{1}{4}$. Also $OB = a\operatorname{cosec}(2\pi\sin\alpha)$ and the length of the geodesic arc AB is $a\tan(2\pi\sin\alpha)$.

Ex. 8. Shew that a first integral of the equations of the geodesics of the cone $u = z\tan\alpha$ is $\sin\alpha\, d\theta = \pm\dfrac{k\,du}{u\sqrt{v^2-k^2}}$, and deduce the equation to the projections of the geodesics in the form

$$u = k\sec(\theta\sin\alpha + \phi),$$

where k and ϕ are arbitrary constants.

Ex. 9. Determine the values of k and ϕ if the geodesic passes through A and B, and deduce the equation (1) of Ex. 4.

250. Geodesics on conicoids. The following theorem is due to Joachimsthal: *If* P *is any point on a geodesic on a central conicoid, r is the central radius parallel to the tangent to the geodesic at* P, *and p is the perpendicular from the centre to the tangent plane to the surface at* P, *pr is constant.*

Let the equation to the conicoid be $ax^2 + by^2 + cz^2 = 1$.

Then at any point of a geodesic,

$$\frac{x''}{ax} = \frac{y''}{by} = \frac{z''}{cz} = \frac{\pm\sqrt{x''^2 + y''^2 + z''^2}}{\sqrt{a^2x^2 + b^2y^2 + c^2z^2}} = \pm\frac{p}{\rho} = \lambda, \text{ say}, \quad \ldots\ldots(1)$$

where ρ is the radius of curvature of the geodesic.

We have also
$$p^{-2} = a^2x^2 + b^2y^2 + c^2z^2,$$
$$r^{-2} = ax'^2 + by'^2 + cz'^2.$$

Whence
$$-p^{-3}p' = a^2xx' + b^2yy' + c^2zz',$$
$$-r^{-3}r' = ax'x'' + by'y'' + cz'z'',$$
$$= \lambda(a^2xx' + b^2yy' + c^2zz'), \text{ by (1)}.$$

Therefore
$$\frac{r'}{r^3} = \lambda\frac{p'}{p^3}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Again, since the tangent to the geodesic is a tangent to the conicoid, $$axx'+byy'+czz'=0,$$

and therefore

$$ax'^2+by'^2+cz'^2=-(axx''+byy''+czz'')$$

or $$r^{-2} \quad =-\lambda(a^2x^2+b^2y^2+c^2z^2), \text{ by (1)},$$

$$=-\lambda p^{-2}. \qquad\qquad (3)$$

Hence, combining (2) and (3),

$$r'p+p'r=0,$$

and therefore pr is constant.

Cor. 1. Since $\lambda=\pm\dfrac{p}{\rho}$, from (3) we deduce $\rho=\pm\dfrac{r^2}{p}$.

(Cf. § 226, Ex. 2.)

Cor. 2. If the constant value of pr is k,

$$\rho=\pm\frac{r^2}{p}=\pm\frac{r^3}{k};$$

hence along a given geodesic the radius of curvature varies as the cube of the central radius which is parallel to the tangent.

Ex. 1. The radius of curvature at any point P of a geodesic drawn on a conicoid of revolution is in a constant ratio to the radius of curvature at P of the meridian section through P.

If α and β are the axes of the meridian section and ρ_1 is its radius of curvature,

$$\rho_1=\frac{\alpha^2\beta^2}{p^3},$$

and we have from § 250, $$\rho=\pm\frac{k^2}{p^3}.$$

Ex. 2. For all geodesics through an umbilic, $pr=ac$.

Ex. 3. Shew that the theorem of § 250 is also true for the lines of curvature of the conicoid.

Ex. 4. The constant pr has the same value for all geodesics that touch the same line of curvature.

Ex. 5. Two geodesics that touch the same line of curvature intersect at a point P. Prove that they make equal angles with the lines of curvature through P.

Ex. 6. PT is the tangent to a geodesic through any point P on the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, and λ, μ are the parameters of the confocals through P. PT makes an angle θ with the tangent to

the curve of intersection of the ellipsoid and the confocal whose parameter is λ. Prove that

$$\lambda\cos^2\theta+\mu\sin^2\theta=\frac{a^2b^2c^2}{k^2},\quad (\text{where } pr=k).$$

The central section parallel to the tangent plane at P, referred to its principal axes, has equation

$$\frac{x^2}{\mu}+\frac{y^2}{\lambda}=1,\ \text{whence}\ \frac{\cos^2\theta}{\mu}+\frac{\sin^2\theta}{\lambda}=\frac{1}{r^2}.$$

We have also $p^2=a^2b^2c^2/\lambda\mu$, and the result immediately follows.

Ex. 7. The tangents to a given geodesic on an ellipsoid all touch the same confocal.

One confocal touches the tangent. Suppose that its parameter is ν. If the normals to the ellipsoid and confocals through P are taken as coordinate axes, the equation to the cone, vertex P, which envelopes the confocal is

$$\frac{x^2}{\nu-\mu}+\frac{y^2}{\nu-\lambda}+\frac{z^2}{\nu}=0.$$

The tangent at P to the geodesic is a generator of this cone, and since its equations are

$$\frac{x}{\cos\theta}=\frac{y}{\sin\theta}=\frac{z}{0},$$

$$\nu=\lambda\cos^2\theta+\mu\sin^2\theta=\text{constant}.$$

Ex. 8. The osculating planes of the geodesic touch the confocal.

251. The curvature and torsion of a geodesic. Consider a geodesic through the origin on the surface

$$2z=\frac{x^2}{\rho_1}+\frac{y^2}{\rho_2}+\dots.$$

If the tangent makes an angle θ with OX, then, at the origin,

$$l_1=\cos\theta,\quad m_1=\sin\theta,\quad n_1=0;$$
$$l_2=0,\quad m_2=0,\quad n_2=1;$$

hence,
$$l_3=\sin\theta,\quad m_3=-\cos\theta,\quad n_3=0.$$

We have, generally,

$$l_2=\frac{-p}{\sqrt{1+p^2+q^2}},\quad m_2=\frac{-q}{\sqrt{1+p^2+q^2}},\quad n_2=\frac{1}{\sqrt{1+p^2+q^2}}.$$

Whence, differentiating l_2 with respect to α, the arc of the geodesic, and applying Frenet's formulae,

$$\frac{l_1}{\rho}+\frac{l_3}{\sigma}=\frac{rl_1+sm_1}{\sqrt{1+p^2+q^2}}+p\frac{d}{d\alpha}(1+p^2+q^2)^{-\frac{1}{2}},$$

which gives, at the origin,

$$\frac{\cos\theta}{\rho}+\frac{\sin\theta}{\sigma}=\frac{\cos\theta}{\rho_1}.$$

Similarly, differentiating m_2, we obtain,

$$\frac{\sin\theta}{\rho}-\frac{\cos\theta}{\sigma}=\frac{\sin\theta}{\rho_2}.$$

Eliminating σ, we have, $\frac{1}{\rho}=\frac{\cos^2\theta}{\rho_1}+\frac{\sin^2\theta}{\rho_2}$, a result obtained in §§ 220, 221.

Eliminating ρ, we have

$$\frac{1}{\sigma}=\sin\theta\cos\theta\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right).$$

Cor. 1. If the surface is developable, so that $\frac{1}{\rho_1}=0$, $\sigma=-\rho\tan\theta$, where θ is the angle at which the geodesic crosses the generator. For a geodesic on a cylinder, θ is constant, and we have the result of § 202.

Cor. 2. If a geodesic touches a line of curvature, its torsion is zero at the point of contact.

Cor. 3. If a geodesic passes through an umbilic, its torsion at the umbilic is zero.

Ex. 1. Shew that $\frac{1}{\sigma^2}=\left(\frac{1}{\rho}-\frac{1}{\rho_1}\right)\left(\frac{1}{\rho_2}-\frac{1}{\rho}\right)$.

Ex. 2. A geodesic is drawn on the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ from an umbilic to the extremity B of the mean axis. Find its torsion at B.

At B, $\rho_1=\frac{a^2}{b}$, $\rho_2=\frac{c^2}{b}$, and therefore

$$\frac{1}{\sigma}=\cos\theta\sin\theta\left(\frac{b}{c^2}-\frac{b}{a^2}\right).$$

Also $pr=ac$, and at B, $p=b$, and

$$\frac{1}{r^2}=\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{c^2};$$

whence $$\frac{1}{\sigma}=\frac{b\sqrt{a^2-b^2}\sqrt{b^2-c^2}}{a^2c^2}.$$

252. Geodesic curvature. Let P, O, C, (fig. 69), be consecutive points of a curve traced on a surface. Along the geodesic through P and O measure off an arc OG equal

to PO, and along PO produced a length OT also equal to PO. PO is ultimately the tangent at O to the curve or geodesic, and the geodesic touches the curve at O. Denote the angle GOT by $\delta\psi_0$, the angle TOC by $\delta\psi$, the angle GOC by $\delta\epsilon$. Then, if $OP = \delta s$,

$$\text{Lt}\,\frac{\delta\psi}{\delta s} = \text{curvature of curve} = \frac{1}{\rho};$$

$$\text{Lt}\,\frac{\delta\psi_0}{\delta s} = \text{curvature of geodesic} = \frac{1}{\rho_0};$$

and $\text{Lt}\,\dfrac{\delta\epsilon}{\delta s}$ is defined to be the **geodesic curvature** of the curve. Let us denote it by $\dfrac{1}{\rho_g}$.

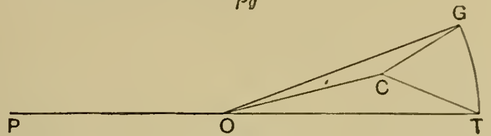

Fig. 69.

The points CGT lie on a sphere whose centre is O, and therefore the arcs CT, TG, GC can be taken to measure the angles COT, TOG, GOC. And since the plane OCG is ultimately that of the indicatrix, and OGT a normal section, the angle CGT is a right angle. Hence,

$$CT^2 = CG^2 + GT^2$$

or

$$\delta\psi^2 = \delta\epsilon^2 + \delta\psi_0^2.$$

Therefore

$$\frac{1}{\rho^2} = \frac{1}{\rho_g^2} + \frac{1}{\rho_0^2};$$

whence the geodesic curvature is expressed in terms of the curvatures of the curve and the geodesic.

Again, if the angle CTG is denoted by ω, ω is ultimately the angle between the planes OCT and OGT, which become respectively the osculating planes of the curve and the geodesic, or the osculating plane of the curve and the normal section of the surface through the tangent to the curve. From the right-angled triangle CGT,

$$\delta\psi_0 = \delta\psi\cos\omega, \quad \delta\epsilon = \delta\psi\sin\omega;$$

whence (i) $\rho = \rho_0\cos\omega$, (ii) $\rho_g = \rho\,\text{cosec}\,\omega = \rho_0\cot\omega$.

We have thus (i) another proof of Meunier's theorem and (ii) the relation between the geodesic and ordinary curvatures of the curve.

Cor. If a curve of curvature ρ^{-1} is projected on a plane through the tangent which makes an angle α with the osculating plane, the radius of curvature of the projection is $\rho \sec \alpha$, (§ 204, Ex. 10). Hence the geodesic curvature of a curve on a surface is the curvature of its projection on the tangent plane to the surface.

Ex. 1. Shew that the geodesic curvatures of the lines of curvature through the origin on the surface

$$2z = \frac{x^2}{\rho_1} + \frac{y^2}{\rho_2} + \frac{1}{3}(ax^3 + 3bx^2y + 3cxy^2 + dy^3) + \ldots$$

are

$$\frac{b\rho_1\rho_2}{\rho_1 - \rho_2}, \quad \frac{c\rho_1\rho_2}{\rho_1 - \rho_2}.$$

Use Ex. 16, § 232.

Ex. 2. Prove that at the origin the geodesic curvature of the section of the surface $ax^2 + by^2 = 2z$ by the plane $lx + my + nz = 0$ is

$$\frac{n(bl^2 + am^2)}{(l^2 + m^2)^{\frac{3}{2}}}.$$

Ex. 3. A curve is drawn on a right cone, semivertical angle α, so as to cut all the generators at the same angle β. Prove that at a distance R from the vertex, the curvature of the geodesic which touches the curve is $\dfrac{\sin^2\beta}{R \tan \alpha}$, and that the geodesic curvature of the curve is $\dfrac{\sin \beta}{R}$.

Ex. 4. By means of the results of Ex. 3 and the result of Ex. 7, § 204, verify the equation $\rho^{-2} = \rho_g^{-2} + \rho_0^{-2}$ for the curve on the cone.

Ex. 5. If u and v are the curvilinear coordinates of a point on a surface and the parametric curves cut at right angles, shew that the geodesic curvatures of the parametric curves are

$$-\frac{1}{\sqrt{GE}}\frac{\partial \sqrt{G}}{\partial u}, \quad -\frac{1}{\sqrt{GE}}\frac{\partial \sqrt{E}}{\partial v}.$$

Consider the curve $U = u$. Let ω be the angle between the osculating plane and the normal section through the tangent. Then the geodesic curvature is given by $\dfrac{1}{\rho_g} = \dfrac{\sin \omega}{\rho}$. Let l_2, m_2, n_2 be the direction-cosines of the principal normal to the curve, then since ω is the complement of the angle between the principal normal to the curve $U = u$ and the tangent to the curve $V = v$,

$$\sin \omega = \frac{l_2 x_u + m_2 y_u + n_2 z_u}{\sqrt{E}}.$$

Now $\qquad \dfrac{l_2}{\rho}=\dfrac{dl_1}{ds}=\dfrac{\partial l_1}{\partial v}\cdot\dfrac{dv}{ds}$; $\quad l_1=\dfrac{x_v}{\sqrt{G}}$; $\quad ds^2=G\,dv^2$.

Therefore
$$\frac{l_2}{\rho}=\frac{1}{\sqrt{G}}\frac{\partial}{\partial v}\left(\frac{x_v}{\sqrt{G}}\right)$$
and
$$\frac{1}{\rho_g}=\frac{1}{\sqrt{EG}}\Sigma\left\{x_u\frac{\partial}{\partial v}\left(\frac{x_v}{\sqrt{G}}\right)\right\}$$
$$=\frac{1}{\sqrt{EG}}\left\{\Sigma\frac{x_u x_{vv}}{\sqrt{G}}+\Sigma x_u x_v\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\right)\right\}.$$

But, since the parametric curves are at right angles, $\Sigma x_u x_v=0$, and therefore
$$\Sigma x_u x_{vv}=-\Sigma x_v x_{uv}=-\frac{1}{2}\frac{\partial}{\partial u}(\Sigma x_v^2)=-\frac{1}{2}\frac{\partial G}{\partial u}.$$

Therefore $\qquad \dfrac{1}{\rho_g}=\dfrac{-1}{2G\sqrt{E}}\dfrac{\partial G}{\partial u}=\dfrac{-1}{\sqrt{GE}}\dfrac{\partial\sqrt{G}}{\partial u}$.

Similarly for the curve $V=v$, $\quad \dfrac{1}{\rho_g}=\dfrac{-1}{\sqrt{GE}}\dfrac{\partial\sqrt{E}}{\partial v}$.

(This solution is taken from Bianchi's *Geometria Differenziale.*)

Ex. 6. If the parametric curves are at right angles and G is a function of v alone and E a function of u alone, the parametric curves are geodesics.

Ex. 7. By means of the expressions given in § 241, Ex. 9, shew that the squares of the geodesic curvatures of the curves of intersection of the ellipsoid $\dfrac{x^2}{a}+\dfrac{y^2}{b}+\dfrac{z^2}{c}=1$ and its onfocals whose parameters are λ and μ, are
$$\frac{a_1b_1c_1}{\lambda(\lambda-\mu)^3},\quad\frac{a_2b_2c_2}{\mu(\mu-\lambda)^3}.$$
Shew how this result may be deduced from that of Ex. 18, § 232.

253. Geodesic torsion. If OT, (fig. 70), is the tangent at O to a curve drawn on a surface, and the osculating plane of the curve makes an angle ω with the normal section through OT, then ω is the angle between the principal normal to the curve and the normal to the surface, and therefore
$$\cos\omega=\frac{-pl_2-qm_2+n_2}{\sqrt{1+p^2+q^2}}.\qquad\qquad(1)$$

The binormal makes an angle $90°\pm\omega$ with the normal to the surface. Let us take as the positive direction of the binormal that which makes an angle $90°-\omega$ with the normal to the surface, and then choose the positive direction of the

tangent to the curve, so that the tangent, principal normal and binormal can be brought into coincidence with **OX**, **OY**, **OZ** respectively. Then

$$\sin\omega = \frac{-pl_3 - qm_3 + n_3}{\sqrt{1+p^2+q^2}}.$$

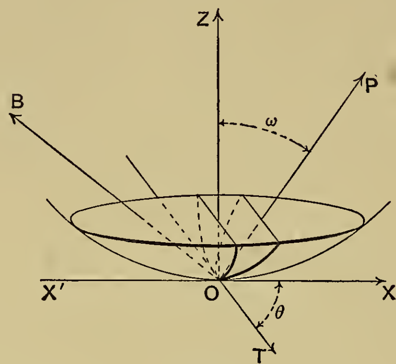

FIG. 70.

Differentiate with respect to s, the arc of the curve, and we have, by (1),

$$\cos\omega\frac{d\omega}{ds} = \frac{\cos\omega}{\sigma} - \frac{l_3(rl_1+sm_1)+m_3(sl_1+tm_1)}{\sqrt{1+p^2+q^2}}$$

$$-(pl_3+qm_3-n_3)\frac{d}{ds}(1+p^2+q^2)^{-\frac{1}{2}}. \quad\text{.........(2)}$$

Now take **O** as the origin, and let the equation to the surface be

$$2z = \frac{x^2}{\rho_1} + \frac{y^2}{\rho_2} + \ldots.$$

Then at the origin (2) becomes

$$\cos\omega\frac{d\omega}{ds} = \frac{\cos\omega}{\sigma} - \frac{l_3l_1}{\rho_1} - \frac{m_3m_1}{\rho_2}. \quad\text{...............(3)}$$

Let **OT** make an angle θ with **OX**.

Then $l_1 = \cos\theta$, $m_1 = \sin\theta$, $n_1 = 0$; and since $n_2 = \cos\omega$,

$$l_3 = m_1n_2 = \sin\theta\cos\omega,$$

$$m_3 = -l_1n_2 = -\cos\theta\cos\omega.$$

Therefore (3) becomes

$$\frac{1}{\sigma}-\frac{d\omega}{ds}=\sin\theta\cos\theta\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right);$$

and hence, by § 251, the value of $\dfrac{1}{\sigma}-\dfrac{d\omega}{ds}$ is the torsion of the geodesic that touches the curve at O. It is called the **geodesic torsion** of the curve, and is evidently the same for all curves which touch OT at O.

Cor. 1. If a curve touches a line of curvature at O its geodesic torsion at O is zero.

Cor. 2. The torsion of a curve drawn on a developable is

$$\frac{\sin\theta\cos\theta}{\rho}+\frac{d\omega}{ds},$$

where θ is the angle at which the curve crosses the generator, ρ is the principal radius, and ω is the angle that the osculating plane makes with the normal section of the surface through the tangent.

Ex. 1. The geodesic torsion of a curve drawn on a surface at a point O is equal to the torsion of any curve which touches it at O and whose osculating plane at O makes a constant angle with the tangent plane at O to the surface.

Ex. 2. The geodesic torsion of a curve drawn on a cone, semi-vertical angle α, so as to cut all the generators at an angle β, is $\dfrac{\sin\beta\cos\beta}{R\tan\alpha}$, where R is the distance of the point from the vertex.

Ex. 3. A catenary, constant c, is wrapped round a circular cylinder, radius a, so that its axis is along a generator. Shew that its torsion at any point is equal to its geodesic torsion, and deduce that

$$\frac{1}{\sigma}=\frac{c\sqrt{z^2-c^2}}{az^2},$$

where z is the distance of the point from the directrix of the catenary.

Examples XIV.

1. A geodesic is drawn on the urface

$$2z=ax^2+2hxy+by^2$$

touching the x-axis. Prove that at the origin its torsion is h.

2. For the conoid $z=f\left(\dfrac{y}{x}\right)$, prove that the asymptotic lines consist of the generators and the curves whose projections on the xy-plane are given by $x^2=cf'\left(\dfrac{y}{x}\right)$, where c is an arbitrary constant.

3. Prove that any curve is a geodesic on its rectifying developable or on the locus of its binormals, and an asymptotic line on the locus of its principal normals.

4. A geodesic is drawn on a right cone, semivertical angle α. Prove that at a distance R from the vertex its curvature and torsion are

$$\frac{p^2}{R^3 \tan\alpha}, \quad \frac{p\sqrt{R^2-p^2}}{R^3 \tan\alpha},$$

where p is the perpendicular from the vertex to the tangent.

5. Prove that the p-r equation of the projection on the xy-plane of a geodesic on the surface $x^2+y^2=2az$ is

$$p^2=\frac{k^2(a^2+r^2)}{k^2+a^2},$$

where k is an arbitrary constant.

6. Prove that the projections on the xy-plane of the geodesics on the catenoid $u=c\cosh\frac{z}{c}$ are given by

$$d\theta=\frac{k\,du}{\sqrt{(u^2-c^2)(u^2-k^2)}},$$

where k is an arbitrary constant.

7. Geodesics are drawn on a catenoid so as to cross the meridians at an angle whose sine is $\frac{c}{u}$, where u is the distance of the point of crossing from the axis. Prove that the polar equation to their projections on the xy-plane is

$$\frac{u-c}{u+c}=e^{2(\theta+\alpha)},$$

where α is an arbitrary constant.

8. A geodesic on the ellipsoid of revolution

$$\frac{x^2+y^2}{a^2}+\frac{z^2}{c^2}=1$$

crosses a meridian at an angle θ at a distance u from the axis. Prove that at the point of crossing it makes an angle

$$\cos^{-1}\frac{cu\cos\theta}{\sqrt{a^4-u^2(a^2-c^2)}}$$

with the axis.

9. Prove that the equation to the projections on the xy-plane of the geodesics on the surface of revolution

$$x=u\cos\theta, \quad y=u\sin\theta, \quad z=f(u)$$

is
$$\theta-\alpha=a\int\frac{\sqrt{1+\left(\frac{dz}{du}\right)^2}}{u\sqrt{u^2-a^2}}\,du,$$

where a and α are arbitrary constants.

10. If a geodesic on a surface of revolution cuts the meridians at a constant angle, the surface must be a right cylinder.

11. If the principal normals of a curve intersect a fixed line, the curve is a geodesic on a surface of revolution, and the fixed line is the axis of the surface.

12. A curve for which $\frac{\rho}{\sigma}$ is constant is a geodesic on a cylinder, and a curve for which $\frac{d}{ds}\left(\frac{\rho}{\sigma}\right)$ is constant is a geodesic on a cone.

13. The curvature of each of the branches of the curve of intersection of a surface and its tangent plane is two-thirds the curvature of the asymptotic line which touches the branch.

14. S_1, S_2, S_3 are the surfaces of a triply orthogonal system that pass through a point O. Prove that the geodesic curvatures at O of the curve of intersection of the surfaces S_2 and S_3, regarded first as a curve on the surface S_2 and then as a curve on the surface S_3 are respectively the principal curvature of S_3 in its section by the tangent plane to S_2 and the principal curvature of S_2 in its section by the tangent plane to S_3.

Verify this proposition for confocal conicoids.

15. Prove that the angles that the osculating planes of the lines of curvature through a point of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ make with the corresponding principal sections are

$$\tan^{-1}\frac{\lambda}{abc}\sqrt{\frac{\mu(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}{(\lambda-\mu)^3}},\ \tan^{-1}\frac{\mu}{abc}\sqrt{\frac{\lambda(a^2-\mu)(b^2-\mu)(c^2-\mu)}{(\mu-\lambda)^3}}$$

where λ and μ are the parameters of the confocals through the point.

INDEX

The numbers refer to pages.

The numbers refer to pages.

The numbers refer to pages.

The numbers refer to pages.

GLASGOW: PRINTED AT THE UNIVERSITY PRESS BY ROBERT MACLEHOSE AND CO. LTD.

Works on Higher Mathematics

An Introduction to the Theory of Infinite Series. By T. J. I'A BROMWICH, M.A., F.R.S. 8vo. 15s. net.

The Modern Theory of Equations. By F. CAJORI. Extra Crown 8vo. 7s. 6d. net.

A Short Course on Differential Equations. By Prof. DONALD F. CAMPBELL, Ph.D. Crown 8vo. 4s. net.

Introduction to the Theory of Fourier's Series and Integrals and the Mathematical Theory of the Conduction of Heat. By Prof. H. S. CARSLAW, M.A. 8vo. 14s. net.

Elliptic Functions. By A. C. DIXON, M.A. Globe 8vo. 5s.

A Course of Plane Geometry for Advanced Students. By CLEMENT V. DURELL, M.A. 8vo. Part I. 5s. net. Part II. 7s. 6d. net.

An Elementary Treatise on Trilinear Co-ordinates, the Method of Reciprocal Polars, and the Theory of Projectors. By Rev. N. M. FERRERS, D.D., F.R.S. Fourth Edition. Crown 8vo. 6s. 6d.

A Treatise on Differential Equations. By A. R. FORSYTH, F.R.S. Fourth Edition. 8vo. 14s. net.

An Elementary Treatise on Curve Tracing. By PERCIVAL FROST, D.Sc., F.R.S. Second Edition. 8vo. 10s. net.

Elementary Treatise on the Calculus. By GEORGE A. GIBSON, M.A., LL.D. Crown 8vo. 7s. 6d.

Elements of Analytical Geometry. By GEORGE A. GIBSON, M.A., LL.D., and P. PINKERTON, M.A., D.Sc. Crown 8vo. 7s. 6d.

LONDON: MACMILLAN & CO., LTD.

Works on Higher Mathematics

Treatise on Bessel Functions. By Prof. A. GRAY and Prof. G. B. MATHEWS. 8vo. 14s. net.

Differential and Integral Calculus. By Sir A. G. GREENHILL, F.R.S. Third Edition. Crown 8vo. 10s. 6d.

Applications of Elliptic Functions. By Sir A. G. GREENHILL, F.R.S. 8vo. 12s.

A Manual of Quaternions. By C. J. JOLY, M.A. 8vo. 10s. net.

An Elementary Treatise on Modern Pure Geometry. By R. LACHLAN, M.A. 8vo. 9s.

A Treatise on the Geometry of the Circle, and some extensions to Conic Sections by the Method of Reciprocation. By W. J. M'CLELLAND, M.A. Crown 8vo. 6s.

The Theory of Determinants in the Historical Order of Development. By THOMAS MUIR, LL.D., F.R.S. 8vo. Vol. I. Part I. General Determinants up to 1841. Part II. Special Determinants up to 1841. 17s. net. Vol. II. The Period 1841 to 1860. 17s. net.

A First Course in the Differential and Integral Calculus. By Prof. WILLIAM F. OSGOOD, Ph.D. Crown 8vo. 8s. 6d. net.

The Theory of Relativity. By L. SILBERSTEIN, Ph.D. 8vo. 10s. net.

Introductory Modern Geometry of Point, Ray, and Circle. By W. B. SMITH, A.M., Ph.D. Crown 8vo. 6s.

An Elementary Treatise on the Theory of Equations. By ISAAC TODHUNTER, F.R.S. Crown 8vo. 7s. 6d.

A Treatise on the Differential Calculus. By ISAAC TODHUNTER, F.R.S. Crown 8vo. 10s. 6d. Key. Crown 8vo. 10s. 6d.

A Treatise on the Integral Calculus and its Applications. By ISAAC TODHUNTER, F.R.S. Crown 8vo. 10s. 6d. Key. Crown 8vo. 10s. 6d.

LONDON: MACMILLAN & CO., LTD.